Abstract—Heat and mass transfer with variable viscosity hydro magnetic flow between two horizontal parallel plates was investigated. In this study we theoretically investigated the effects of viscosity variation parameter, magnetic field parameter and energy dissipation parameter. We considered steady incompressible fluid flow between two horizontal parallel plates which allow energy dissipation. The equation(s) governing the flow were derived and solved. The solution were extended in order to examine the flow structure as λ increases, the solutions were computed for wall skin friction, material and heat flux in powers of λ for (λ≥0,n₁≥0,n₂≥0 ). The results obtained were presented using tables and graphs. It was noted that the usual parabolic Poiseuille was observed with minimum values at the wall and maximum values at the plates’ centerline. At low magnetic field intensity, the magnitude of skin friction, wall heat and material flux increases with increase in fluid viscosity. The rate of material transfer to the wall decreases with an increase in fluid viscosity and decrease in magnetic field intensity.

Keywords—Energy Dissipation; Heat and Mass Transfer; MHD; Steady Flow; Variable Viscosity.

I. INTRODUCTION

Parallelo-POISEUILLE flow constitutes a simple class of parallel flows in fluid dynamics with many applications in mathematical modeling of several physical and engineering systems. This type of flow normally occurs between parallel stationary plates due to an imposed constant axial pressure gradient. Its general characteristic is a parabolic axial velocity profile. Moreover, the viscosity of many fluids varies with temperature and concentration, for example blood, glycerin syrup, various lubricants used in engineering systems like polymer solutions, mineral oils with polymeric additives, among others. This variation in fluid viscosity will certainly affect the flow characteristics. Sliding friction is greatly reduced if a fluid is forced between two solid surfaces moving relative to each other. Practical examples include: Hydrostatic thrust bearing and air-cushioned vehicles. However, it has been established that the engineering systems in general are greatly affected by the application of an external magnetic field. Our results showed that at low magnetic field intensity, a turning point exists in the flow field and two solution branches can be observed.

1.1. Definition of Terms

Porous Media: Is a material containing pores (voids) which allow the passage of matter (fluid).

Viscosity: Is the property of a fluid which determines its resistance to shearing stresses (It is the measure of the internal fluid friction which causes resistance to flow).

Porous Plates: Represents a material through which fluid is forced into or out. It has the pores that allows the fluid to pass through.

Boundary Layer: In physics and fluid mechanics a boundary layer is that layer of fluid in the immediate vicinity of a bounding surface where effects of viscosity of the fluid are felt. Convective heat transfer: Is the transfer of heat from one place to another by movement of fluid.

Horizontal Channel: Two parallel horizontal trend lines acting as a very strong support and resistance.

Heat: May be defined as energy in transit from a high temperature object to a lower temperature object.
Mass: Is the amount matter in an object.
Energy Dissipation: As a charge q moves through a resistor, it loses potential energy qV where V is the potential drop across the resistor. This energy goes into heat, much like the way a ball of putty that falls off a cliff converts its potential energy to heat when it hits the ground. We refer to this conversion of potential energy into heat as dissipation.
Parallel Plates: These represent definite parallel flat walls which restrict the flow of fluid.
MHD Flow (Magneto Fluid Dynamics or Hydro-magnetics): Is an academic discipline which studies the dynamics of electrically conducting fluids.
Bifurcation: Is a period-doubling, a change from an N-point attractor, which occurs when a control parameter is changed.

II. Literature Review

The two dimensional steady hydro-magnetic flow of a Newtonian electrically conducting viscous incompressible and radiating fluid between two parallel plates has been considered by many engineers and scientists.

Govinda et al., [3] studied hydro-magnetic laminar flow through conducting parallel porous plate. Here the porosity of wall was taken into account. In their study consideration was also given to the rectangular channel when there is current and when there is no current and they found that the general solution had two unknown constants. By choosing the constants they showed that the general solution can be made to fit the solution of two dimensional channels whose geometry approaches, in the limit to that of one dimensional channel.

Vdyamidhi & Rao [13] studied two dimensional unsteady flow of a conducting viscous incompressible fluid between two parallel porous plates one of which is fixed while the other is uniformly accelerated when there is transverse magnetic field. They found out that for a given Hartman number “m” a suction parameter B increase in the velocity at any point of the fluid increases the skin friction and the stationary plate increases while that at the accelerated plate decreases.

Kearsley [5] studied problem of steady state coutte flow with viscous heating. In his study an exact solution was found for the non-linear problem with mechanical coupling. The steady flow of a fluid with viscosity exponentially thermo stated dependent on temperature which he shared between an adiabatic fixed inner cylinder and rotating outer cylinder. There was maximum torque above which no steady flow is possible and below which two flows are possible, a high shear and a low shear steady flow for each value or torque.

Raptis et al., [12] analyzed the problem of hydro-magnetism free convection flow through porous medium between parallel plates. In their study the effect such as buoyancy boundary and inertia of porous media, Hartman effect MHD and heat generation or absorption of fluid. Similarities valuable were employed for the case of variable surface temperature and resulting ordinary differential equations were solved.

Bhargava & Takhar [1] studied the numerical solution of free convection MHD micro polar fluid between two parallel porous plates. The basic of electromagnetic induction were briefly reviewed and the magnetic Reynolds’s number which quantifies induction relative to losses due to ohmic resistance was defined. They studied in the present of temperature dependent heat source including the effect of friction heating in the presence magnetic field profile for the velocity. Micro rotation and temperature are presented for a wide range of Hartmann numbers and micro polar parameter.

Makinde & Mbore [8] investigated the effect of thermal radiation on MHD oscillatory flow in a channel filled with saturated porous medium, and non-uniform wall temperatures. Their result shows that increasing magnetic field intensity reduces wall shear stress while increasing radiation parameter through heat absorption causes an increase in magnitude of the wall shear stress.

Okelo [10] investigated unsteady free convection incompressible fluid past a semi-infinite vertical porous plate in the present of a strong magnetic field inclined at an angle to the plate with hall and ion slip effects. He found that an increase in mass diffusion parameters causes an increase in the concentration profile, an increase in the angle of inclination in Eckert number (Ec) causes an increase in temperature profile and also an increase in the angle of inclination lead to an increase in primary velocity profile and a decrease in secondary velocity profile.

Guria [4] studied hydro-magnetic flow between two porous disks rotating about non coincident axes. He found out that the temperature increases with increase in either M2 or K2. He found that the rate of heat transfer of the disk increases with an increase in temperature.

Kumar et al., [7] considered the problem of unsteady MHD periodic flow of viscous fluid through a planar channel in porous medium using perturbation techniques. In their study the governing equations have been solved by perturbation technique. The analytical expression for velocity profiles of the fluid with numerical computations involved were presented. The effect of various parameters on the flow field have been discussed. The increasing values of radiation parameter N showed similar effect on the velocity profile. The fluid velocity decreases as the values of Eckert number increases. The flow decelerates on the imposition of the magnetic field that is on the increase of the magnetic parameter.

Ismail Gboye Gaoku et al., [2] studied magnetic field and thermal effect on steady hydro-magnetic coutte flow through a porous channel. In their study they highlighted the effect of magnetic field, radiation and permeability parameters on both profiles.

Israel Cokey et al., [11] investigated the combined effects of thermal radiation and transverse magnetic field on the study flow of electrically conducting optically thin fluid through a horizontal channel filled with saturated porous medium and non-uniform wall temperature. Their results showed that an increasing magnetic field radiation parameters reduced the velocity and temperature profiles as well as the
shear stress at the wall, while increasing radiation parameter causes an increase in the magnitude of the rate of heat transfer.

Narahari [9] studied the effect of thermal radiation and free convective currents on unsteady coutte flow between two vertical parallel plates with constant heat flux at one boundary. It was observed that the velocity increases for both the impulsive motion as well as the accelerated motion of one of the walls with an increase in radiation parameter. It is also observed that the velocity decreases with an increase in either Prandtl number or time for both the impulsive motion as well as the accelerated motion. An increase in Grashof number leads to a fall in velocity due to enhancement in buoyancy force. An increase in the radiation parameter leads to increase in temperature of the flow field. Further, the shear stress at the moving wall increases for both the impulsive motion as well as the accelerated motion of one of the walls with an increase in radiation parameter. The rate of heat transfer increases with an increase in either radiation parameter or Prandtl number while it decreases with an increase in time.

Kinyanjui et al., [6] studied the effect of hall current and rotation parameter on the dissipative fluid flow past vertical semi-infinite plates. It was observed that an increase in hall parameter for both cooling of the plate by free convection currents and heating of the plate by free convection currents has no effect on the temperature profiles, an increase in rotational parameter led to a decrease in the velocity profile.

In our study, we considered the effect of variable viscosity on heat and mass transfer in hydro-magnetic flow between two horizontal parallel plates with energy dissipation. We investigated the bifurcation that takes place in the flow fields as the viscosity parameters increases with respect to magnetic field intensity and energy dissipation parameters.

III. MATHEMATICAL FORMULATION

In this section Heat and mass transfer in variable viscosity hydro-magnetic steady flow between two parallel horizontal plates with energy dissipation is considered. The equation(s) governing the flow are generated and solve for various parametric conditions.

Consider flow through a two dimensional parallel plates under the influence of an externally applied homogeneous magnetic field. It is assumed that the fluid is incompressible and has a small electrical conductivity and the electromagnetic force produced is also very small. We choose a Cartesian coordinate system (x,y) where 0x lies along the stream wise direction, y is the distance measured transversely such that y=±a are the channel walls, let u and v be the velocity components in the directions of increasing x and y respectively

The continuity, Navier-stokes, energy and concentration equations governing the flow simplify to:

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]  

\[ \rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} - \sigma_\Omega \beta_0 \mu \]  

\[ \rho \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = - \frac{\partial p}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} \]  

\[ u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \kappa \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \frac{\mu}{\rho C_p} \left( \frac{\partial u}{\partial x} \right)^2 \left( \frac{\partial v}{\partial y} \right)^2 \]  

\[ \frac{\partial C}{\partial x} + \frac{\partial C}{\partial y} = D \frac{\partial^2 C}{\partial x^2} \]  

Where \( \tau_{xx}, \tau_{yy} \) and \( \tau_{xy} \) are the stress components, i.e.

\[ \tau_{xx} = \mu \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \]  

\[ \tau_{yy} = -2 \mu \frac{\partial v}{\partial y}, \tau_{xy} = -2 \mu \frac{\partial u}{\partial x} \]  

And \( \mu \) is the dynamic coefficient of viscosity defined as

\[ \mu = \mu_0 \left[ 1 + \alpha (T - T_0) \right] \beta (C - C_0) \]  

Also \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \), \( BO = \mu_e H_0 \) is the electromagnetic induction, \( \mu_e \) the magnetic permeability, \( H_0 \) the intensity of magnetic field, \( \mu \) the conductivity of the fluid, \( P \) the pressure, \( T \) the temperature, \( C \) the concentration, \( \rho \) the density, \( K \) the coefficient of thermometric conductivity, \( D \), the coefficient of mass diffusivity. \( \alpha \) and \( \beta \) the coefficients of viscosity variation due to temperature and concentration respectively, and \( \mu_o, T_o \) and \( C_o \) are the reference viscosity, temperature and concentration respectively.

In order to construct a simple mathematical model that shows the effects of viscosity variation on channel flow, the following assumptions are made:

(i) The normal velocity is assumed to be zero, while the axial velocity is dependent of axial distance, i.e.

\[ v = 0, u = u(y) \]
(ii) We assume uniform axial temperature and concentration wall gradients together with uniform axial pressure gradient, i.e.

$$ T = Ax + r(y), C = Bx + z(y), \quad \frac{\partial p}{\partial x} = \text{Constant} \quad (10) $$

where A and B are constants.

(iii) In the governing equations, we retain the variations in rate of change of viscosity where it occurs as $\frac{\partial \mu}{\partial x}$ and replace $\mu$ by its mean value $\mu_0$. This amounts to assuming that $\alpha (r - r_0) << 1$ and $\beta (C - C_0)$ which is justified in most real problems.

Using (8) and the assumptions (i to iii) on the governing equations, we obtain:

$$ \frac{d^2 u}{dy^2} - \frac{1}{\mu} \frac{\partial p}{\partial x} = - \frac{1}{\mu} \frac{\partial \mu}{\partial y} \frac{du}{dy} + \frac{\sigma e B_0}{\mu_0} \mu \quad \frac{\partial p}{\partial y} = \frac{\partial \mu}{\partial y} \frac{du}{dy} \quad (11) $$

$$ \frac{2}{\mu} \frac{d}{dy} \left( \frac{\partial p}{\partial y} \frac{du}{dy} \right) = \frac{\partial \mu}{\partial y} \frac{du}{dy} \quad (12) $$

$$ \frac{2}{\mu} \frac{d^2 \phi}{dy^2} = \frac{\partial \mu}{\partial y} \frac{du}{dy} \quad (13) $$

From equations (12), (13) and (14) we rewrite equation (11) as

$$ \frac{d^2 u}{dy^2} - \frac{1}{\mu_0} \frac{\partial \mu}{\partial y} \frac{du}{dy} + \frac{\sigma e B_0}{\mu_0} \mu \frac{\partial \mu}{\partial y} \frac{du}{dy} \quad (14) $$

i.e. a non-linear integral-differential equation that governs the axial velocity component. It is important to note that in the derivation of equation (15), we require:

$$ \frac{dr}{dy} = 0, \frac{dz}{dy} = 0 \text{ at the walls.} \quad (16) $$

The following dimensionless variables and parameters are introduced:

$$ u = \frac{1}{U}, x = \frac{1}{a}, y = \frac{1}{a}, r = \frac{1}{a} \frac{r}{r_0}, z = \frac{1}{z}, \frac{\partial p}{\partial y} = \frac{1}{\mu_0} \frac{\partial \mu}{\partial y} \frac{du}{dy} \quad (17) $$

$$ \lambda = \left( \frac{\alpha A}{K} + \frac{\beta B}{D} \right) \frac{3}{\mu_0} \frac{hU_n}{n_1}, \quad n_1 = \frac{\sigma e B_0}{\lambda \mu_0}, \quad n_2 = \frac{2}{\lambda K e C_p} \frac{hU_n}{n_1}, $$

$$ \frac{\partial u}{\partial y} = \frac{1}{2} \left( \frac{2}{y - 1} + \frac{\lambda}{360} \right) \left[ n_1 y^{-4} - 2 y^{-4} + 15 y^{-1} + \frac{13}{2} y^{-4} + \frac{14}{y - 1} \right] $$

$$ \frac{\partial u}{\partial y} = \frac{1}{2} \left( \frac{2}{y - 1} \right) \left[ n_2 y^{-2} + 4459 y^{-2} - 42885 n_1 y^{-1} + 2184 n_2 y^{4} + 8145 n_1 y^{4} + 1729 y^{4} + \ldots \right] + 0(\lambda) \quad (23) $$

Where $h = \frac{\partial p}{\partial x}$ the uniform pressure gradient and $\lambda$, $n_1$, and $n_2$ is are viscosity variation parameter, magnetic field parameter and energy dissipation parameter respectively. Equation (15) remains the same with dimensionless variables; hence we neglect the primes for clarity. Letting $u = \frac{1}{h}$, the dimensionless form of the non-linear integral-differential equation that governs the axial velocity component then becomes:

$$ \frac{d^2 u}{dy^2} - \frac{1}{\mu_0} \frac{\partial \mu}{\partial y} \frac{du}{dy} + \frac{\sigma e B_0}{\mu_0} \mu \frac{\partial \mu}{\partial y} \frac{du}{dy} \quad (18) $$

The appropriate boundary conditions are $\frac{\partial u}{\partial y}(y = 0) = 0$. Once the problem (18) has been solved for $\frac{\partial u}{\partial y}(y)$, then equations (13) and (14) can be easily integrated with respect to $y$ from $y = 0$ to $y = 1$, i.e.

$$ H = \frac{\rho C_p}{h \mu_0} \frac{K \partial}{dy} = \frac{S}{h \mu_0} \frac{du}{dy} \quad (19) $$

where $S = \frac{\rho C_p}{h \mu_0} \frac{K \partial}{dy}$, and

$$ G = \frac{C_0 D}{h \mu_0} \frac{dz}{dy} = \frac{1}{a} \frac{du}{dy} \quad (20) $$

The skin friction is given by:-

$$ \tau_w = \frac{\partial u}{\partial y}(1) \quad (21) $$

The right hand sides of equations (19) and (20) are proportional to the temperature and concentration gradients normal to the boundaries at $y = \pm 1$.

**IV. Method of Solution**

The non-linear nature of equation (18) precludes its solution exactly, hence, we seek the solution as an asymptotic power series in terms of the viscosity variation parameter $\lambda$, i.e.

$$ u = \sum_{i=0}^{n} \lambda^i A_i \quad (22) $$

We substitute the above expression (22) into (18) and collect the coefficients of like powers of $\lambda$. The resulting equations governing $A_i$ along with their corresponding boundary conditions are solved iteratively. Since it is very cumbersome to obtain manually many terms of the solution series (which are necessary in order to improve the resulting series solution using pde approximants technique). Using the pde approach techniques you get:-

$$ u = \frac{1}{2} \left( \frac{2}{y - 1} + \frac{\lambda}{360} \right) \left[ n_1 y^{-4} - 2 y^{-4} + 15 y^{-1} + \frac{13}{2} y^{-4} + \frac{14}{y - 1} \right] $$

$$ \frac{\partial u}{\partial y} = \frac{1}{2} \left( \frac{2}{y - 1} \right) \left[ n_2 y^{-2} + 4459 y^{-2} - 42885 n_1 y^{-1} + 2184 n_2 y^{4} + 8145 n_1 y^{4} + 1729 y^{4} + \ldots \right] + 0(\lambda) \quad (23) $$
4.1. Series Summation and Improvement

We extend the solution series using computer symbolic algebra package in order to examine the flow structure as \( \lambda \).

\[
\tau_w = 1 + \frac{\lambda}{15} (n_2 - 5) + 2 \frac{\lambda}{2835} (63n_2 - 387n_1 + 2) + 0(\lambda^3)
\]

\[
G = -\frac{1}{3} + \frac{\lambda}{15} \left( \frac{1}{35} - \frac{n_2}{105} + \frac{2}{15} m \right) + \lambda \left( \frac{11}{315} n_1 + \frac{m n_2}{63} - \frac{17}{315} n_1 - \frac{2}{15} \frac{2}{105} - \frac{1}{35} \right) + 0(\lambda^3)
\]

\[
H = -\frac{1}{3} (1 + S) + \lambda \left( \frac{4}{15} - \frac{n_2}{105} - \frac{2}{35} + S \left( \frac{2}{15} \frac{n_2}{105} - \frac{1}{35} \right) \right) + 0(\lambda^3)
\]

The first 15 coefficients for the above series representing the flow characteristics were obtained. We recast the series into several diagonal, Pde' approximants in order to improve their usefulness. For instance, the series for the wall skin friction is transformed as:

\[
\tau_w = \sum_{i=0}^{N} C_i \lambda^i = \sum_{i=0}^{N} M_i \frac{a_i \alpha_i \lambda^i}{b_i \beta_i \lambda^i}
\]

Where \( N = 2M \) is the order of the series required for each approximant. The idea is to match the Taylor expansion as far as possible. This method is good at evaluating functions beyond the radius of convergence of the corresponding infinite series. It fails when we are evaluating near the zeros of the denominator of the fraction. We compute the turning points in the flow fields, especially for the case of low magnetic fields intensity with and without energy dissipation i.e., \( n_2 \geq 0 \) and the results are as shown in Tables 2 and 3.

4.2. Variable Viscosity Hydro-magnetic Couette Flow

We investigate the combined effects of a temperature and concentration dependent fluid viscosity and an externally applied homogeneous magnetic field on the flow of a viscous incompressible fluid placed between two infinite parallel plates moving relative to each other with constant velocity. Using the boundary condition \( u(a) = U \), in (9), the integral-differential equation (18) is solved using the perturbation method to give some terms of the solution series for the axial velocity component as:

\[
u = \frac{1}{2} \left( \frac{y^2}{234} + \frac{\lambda^2}{275600} \right) \left( -8y^3 + 16n_2 y^2 + 64n_1 y + 124n_2 - 47y + 180n_1 + 164n_2 - 47y + 17n_2 + 180n_1 - 47 \right) + 0(\lambda^3)
\]

\[
G = \lambda \left( \frac{43}{4032} \frac{m_1}{30} \frac{231}{20160 n_2} + \frac{2}{15} \frac{599}{212896} + \frac{1633}{241920} n_1 m_2 + \frac{2}{15} \frac{624823}{15967200 n_2^2} - \frac{2}{15} \frac{463}{241920} n_1 m_2 + \frac{2}{15} \frac{67}{20160 n_1} - \frac{2}{15} \frac{9009}{26611200 n_2^2} \right) + 0(\lambda^3)
\]

\[ H = \lambda \left( \frac{55}{12} - \frac{13}{15} \right) \left( \frac{43}{4032} - \frac{371}{20160 n_2} + \frac{2}{15} \frac{599}{212896} \frac{2}{15} \frac{1633}{241920} n_1 m_2 + \frac{2}{15} \frac{624823}{15967200 n_2^2} - \frac{2}{15} \frac{463}{241920} n_1 m_2 + \frac{2}{15} \frac{67}{20160 n_1} - \frac{2}{15} \frac{9009}{26611200 n_2^2} \right) + 0(\lambda^3)
\]

V. DISCUSSION

In this section, the solutions for the heat and mass transfer in variable viscosity hydro-magnetic flow between two parallel plates with energy dissipation were presented graphically and discussed for various parametric conditions.

5.1. Graphical Representation of Results

From equation (23) we generate figures 2-4.
Figure 2: Axial Velocity Profile, $\lambda=0.5, n_2=0.5$

Figure 2 shows the axial velocity profile of the flow. The usual parabolic Poiseuille profile is observed with minimum value at the walls and absolute maximum value at the plate’s centerline. However, an increase in magnetic field intensity (i.e. $n_1=0, 0.5, 1.0$) causes a general decrease in the magnitude of axial velocity.

Figure 3: Axial Velocity Profile $\lambda=0.5, n_1=0.5$

Figure 3 shows the axial velocity profile of the flow. The usual parabolic Poiseuille profile is still observed with minimum value at the walls and absolute maximum value at the plate’s centerline. However, an increase in the energy dissipation intensity (i.e. $n_2=0, 2, 4$) causes a general decrease in axial velocity.

Figure 4: Axial Velocity Profile, $n_1=0.5, n_2=0.5$

Fluid velocity also decreases with an increase in the fluid viscosity (for $\lambda=0.1, 0.3, 0.5$). The graphs (3.05) to (3.07) are generated by equations (24), -(26) respectively.

Figure 5: Skin Friction, $n_2=0.5$

At very low magnetic field intensity, the magnitude of skin friction increases with increase in the fluid viscosity as shown in figure 5. In the presence of a magnetic field, a general decrease in the magnitude of skin friction across the wall is observed. Increase in magnetic field intensity causes a further decrease in the magnitude of skin friction.

Figure 6: Material Flux, $n_2=0.5$

At very low magnetic field intensity, the magnitude of wall heat flux increases with increase in the fluid viscosity as shown in figure 6. In the presence of a magnetic field, a general decrease in the magnitude of heat transfer across the wall is observed. Meanwhile an increase in magnetic field intensity causes a further decrease in the magnitude of skin friction, wall heat transfer.

Figure 7: Wall Heat Flux, $n_2=0.2$

At very low magnetic field intensity, the magnitude of material flux increase with increase in the fluid viscosity as shown in figures 7. In the presence of a magnetic field, a general decrease in the magnitude of material transfer across the wall is observed. Meanwhile an increase in magnetic field intensity causes a further decrease in the magnitude of material transfer across the wall. 
intensity causes a further decrease in the magnitude material transfer across the wall.

Figures 8-11 are generated from equation (28)-(31) respectively.

**Figure 8: Axial Velocity Profile, \( \lambda = 0.5, n_2 = 0.5 \)**

Figure 8 shows the axial velocity profile of the flow. Generally, the fluid velocity is maximum at the moving wall and decreases to zero velocity at the stationary wall. However, an increase in magnetic field intensity (i.e. \( n_1 = 0, 0.5, 1 \)) causes a decrease in the fluid velocity.

**Figure 9: Plot of Skin Friction \( n_2 = 0.5 \)**

Figure 9 shows the variation of the wall skin friction with respect to viscosity variation parameter (\( \lambda \)). The wall friction increases in the magnitude as the fluid viscosity increases, a further increase is observed with an increase in the magnetic field intensity (\( n_1 \)). Similar effect is observed with an increase in viscous dissipation parameter (\( n_2 \)).

**Figure 10: Heat Flux at \( y=1, n_2 = 0.5, S = 1 \)**

The rate of heat flux to the wall decreases with an increase in the fluid viscosity as shown in figure 10 an increase in magnetic field intensity causes a further decrease in the heat flux across the wall.

**Figure 11: Material Flux \( y=1, n_2 = 0.5 \)**

The rate of material transfer to the wall decreases with an increase in the fluid viscosity as shown in figure 11 an increase in magnetic field intensity causes a further decrease in the material flux across the wall.

### VI. Results

**Table 1: Numerical Results showing some Coefficient of the Skin Friction (\( \tau_w \))**

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<td>-0.6278231026x10^-7</td>
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<tr>
<td>14</td>
<td>0.3538938679922x10^-8</td>
<td>0.79132652273x10^-9</td>
<td>0.79132652273x10^-9</td>
</tr>
<tr>
<td>15</td>
<td>0.10044560016018x10^-10</td>
<td>-0.5319890137728x10^-10</td>
<td>-0.5319890137728x10^-10</td>
</tr>
</tbody>
</table>
Table 1: Shows that the singularity lies on the positive axis. It is very interesting to note that the magnitude of the turning point decreases with an increase in the energy dissipation and increases with increase in the magnetic field parameter as shown in tables 2 and 3.

Table 2: Numerical Results showing the Turning Point ($\lambda_c$) for $n_1 \geq 0$, $n_2 = 0$

<table>
<thead>
<tr>
<th>$n_2$</th>
<th>0</th>
<th>0.5</th>
<th>1.0</th>
<th>1.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_c$</td>
<td>3.8754</td>
<td>3.2208</td>
<td>2.7958</td>
<td>2.4658</td>
</tr>
</tbody>
</table>

Table 3: Numerical Results showing the Turning Point ($\lambda_c$) for $n_1 \geq 0.5$, $0 < n_2 < 1$

<table>
<thead>
<tr>
<th>$n_1$</th>
<th>0</th>
<th>0.001</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_c$</td>
<td>3.2208</td>
<td>3.2326</td>
<td>3.3443</td>
</tr>
</tbody>
</table>

Table 4: Numerical Results showing some Coefficients of Skin Friction ($\tau_{\theta\theta}$)

<table>
<thead>
<tr>
<th>i</th>
<th>$C[i]n_1=0.0,n_2=0.5$</th>
<th>$C[i]n_1=0.5,n_2=0.5$</th>
<th>$C[i]n_1=0.5,n_2=0.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.5000000000000</td>
<td>1.5000000000000</td>
<td>1.5000000000000</td>
</tr>
<tr>
<td>1</td>
<td>0.0642361111111</td>
<td>0.2100694444444</td>
<td>0.0203188888888</td>
</tr>
<tr>
<td>2</td>
<td>0.0021499875999</td>
<td>0.000287801752</td>
<td>-0.00083719135</td>
</tr>
<tr>
<td>3</td>
<td>0.0000425773300</td>
<td>0.001361410085</td>
<td>0.000064392489</td>
</tr>
<tr>
<td>4</td>
<td>-0.68156739031510^{-6}</td>
<td>-0.00015959642</td>
<td>-0.38792745910^{-5}</td>
</tr>
<tr>
<td>5</td>
<td>0.11686696069910^{-6}</td>
<td>0.000027163236</td>
<td>0.2101964114910^{-6}</td>
</tr>
<tr>
<td>6</td>
<td>-0.9137157719010^{-6}</td>
<td>-0.4660505798610^{-6}</td>
<td>-0.10667030710^{-7}</td>
</tr>
<tr>
<td>7</td>
<td>0.78703949368910^{-6}</td>
<td>0.85817704342710^{-6}</td>
<td>0.513214098010^{-7}</td>
</tr>
<tr>
<td>8</td>
<td>-0.6685648913810^{-10}</td>
<td>-0.15187961699710^{-9}</td>
<td>-0.234532912410^{-10}</td>
</tr>
<tr>
<td>9</td>
<td>0.5773257135810^{-10}</td>
<td>0.285252517710^{-10}</td>
<td>0.10104387130610^{-11}</td>
</tr>
<tr>
<td>10</td>
<td>-0.5040478301510^{-12}</td>
<td>-0.546275765010^{-10}</td>
<td>-0.14117368205810^{-11}</td>
</tr>
<tr>
<td>11</td>
<td>0.44477514713410^{-13}</td>
<td>0.1063044499810^{-9}</td>
<td>0.14117368205810^{-14}</td>
</tr>
<tr>
<td>12</td>
<td>-0.3960342844410^{-14}</td>
<td>-0.20964782316010^{-6}</td>
<td>-0.378428186410^{-16}</td>
</tr>
<tr>
<td>13</td>
<td>0.35544338184710^{-15}</td>
<td>0.418145723200010^{-10}</td>
<td>0.203471390910^{-18}</td>
</tr>
<tr>
<td>14</td>
<td>-0.3212476702810^{-16}</td>
<td>-0.4207580638810^{-11}</td>
<td>-0.72010625723010^{-19}</td>
</tr>
<tr>
<td>15</td>
<td>0.2921450442910^{-17}</td>
<td>0.1709974731210^{-11}</td>
<td>0.734793072610^{-20}</td>
</tr>
</tbody>
</table>

Table 5: Numerical Results showing the Nearest Singularity in the Series i.e. The Radius of Convergence ($n_2 = 0.5$)

<table>
<thead>
<tr>
<th>$n_1$</th>
<th>0</th>
<th>0.1</th>
<th>0.3</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_c$</td>
<td>-9.9842</td>
<td>-7.3486</td>
<td>-5.4897</td>
<td>-4.4107</td>
</tr>
</tbody>
</table>

Table 6: Numerical Results showing the Turning Point ($\lambda_c$) for $n_1 \geq 0.5$, $0 < n_2 < 1$

<table>
<thead>
<tr>
<th>$n_1$</th>
<th>0</th>
<th>0.001</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_c$</td>
<td>3.2208</td>
<td>3.2326</td>
<td>3.3443</td>
</tr>
</tbody>
</table>

Table 7: Numerical Results showing some Coefficients of Skin Friction ($\tau_{\theta\theta}$)

<table>
<thead>
<tr>
<th>i</th>
<th>$C[i]n_1=0.5,n_2=0.5$</th>
<th>$C[i]n_1=0.5,n_2=0.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.5000000000000</td>
<td>1.5000000000000</td>
</tr>
<tr>
<td>1</td>
<td>0.0642361111111</td>
<td>0.2100694444444</td>
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<td>2</td>
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<tr>
<td>3</td>
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<td>0.001361410085</td>
</tr>
<tr>
<td>4</td>
<td>-0.68156739031510^{-6}</td>
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<td>7</td>
<td>0.78703949368910^{-6}</td>
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<tr>
<td>8</td>
<td>-0.6685648913810^{-10}</td>
<td>-0.15187961699710^{-9}</td>
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<td>0.5773257135810^{-10}</td>
<td>0.285252517710^{-10}</td>
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<tr>
<td>10</td>
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<td>15</td>
<td>0.2921450442910^{-17}</td>
<td>0.1709974731210^{-11}</td>
</tr>
</tbody>
</table>

VII. Summary

In our study we derived the equations governing heat and mass transfer in variable viscosity MHD steady flow between two parallel horizontal plates with energy dissipation which were nonlinear integral-differential equation in chapter three. The resulting equations along with their corresponding boundary conditions were solved iteratively, the results were presented in tables and graph and discussed in chapter four.

VIII. Conclusion

It was found that an increase in fluid viscosity led to an increase in the value of the parameter $\lambda$ due to an increase in additive concentration and decrease in fluid temperature. The result gives usual parabolic Poiseuille profile for axial velocity profile, with minimum value at the walls and maximum values at the plates’ centerline, generally the fluid velocity is maximum at the moving wall and decreases to zero at the stationary wall. However an increase in magnetic field intensity $n_1$ causes a general decrease in the magnitude of axial velocity, similar effect is observed with a decrease in the energy dissipation intensity. Fluid velocity also increases with a decrease in fluid viscosity. Generally in the presence of a magnetic field, a general decrease in the magnitude of skin friction, heat and material flux is observed.

IX. Recommendations

This piece of work considered heat and mass transfer variable viscosity hydro-magnetic steady flow between two horizontal parallel plates with energy dissipation. Our results show that a decrease in the fluid viscosity and an increase in the viscous dissipation can cause an increase in the fluid velocity, a decrease in the magnitude of the wall skin friction and an increase in the magnitude of heat and material flux across the wall. The above model is therefore appropriate to simulate wind tunnel tests of lubrication phenomenon in engineering systems. We therefore recommend that this work can be extended by considering the following:

(i) Unsteady compressible fluid
(ii) Experimental investigation of the same problem.
REFERENCES


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Albert Aluoch Kalal B.Ed.(Science) Mathematics/Chemistry M’Sc(Applied Mathematics)

