Finite Difference Solution of Seepage Equation: 
A Mathematical Model for Fluid Flow


*Department of Pure and Applied Mathematics, Jomo Kenyatta University of Agriculture and Technology, Nairobi, KENYA. E-Mail: nelsonnyachwaya [at] gmail [dot] com

**Department of Pure and Applied Mathematics, Jomo Kenyatta University of Agriculture and Technology, Nairobi, KENYA. E-Mail: jksigey [at] jkuat [dot] ac [dot] ke

***Department of Pure and Applied Mathematics, Jomo Kenyatta University of Agriculture and Technology, Nairobi, KENYA. E-Mail: jokelo [at] jkuat [dot] ac [dot] ke

****School of Mathematics, University of Nairobi, Nairobi, KENYA. E-Mail: jmkwoyo [at] uonbi [dot] ac [dot] ke

Abstract—The third order seepage parabolic partial differential equation (which models the fluid flows) was solved by two types of finite differences methods, which are Alternating Direction Explicit (ADE) method and Alternating Direction Implicit (ADI) method subject to some boundary and initial conditions. We compared the numerical results for the two methods. First, we derived the finite differential form of ADE and ADI methods for the given model and then presented an algorithm for each method. The resulting systems of linear algebraic equations were solved using Mathematica software. The solutions were represented graphically in three dimensions and interpreted. We also studied the numerical stability of both methods by matrix Method. We observed that the seepage flow decreased with distance from the source (dam) and also the smaller the mesh sizes, the finer the decrease in the fluid seepage. Seepage is a very slow process. Fluid seeps few meters in several years. That is why the slope of the surfaces decreases slowly.

Keywords—Fluid Flow; Seepage; Parabolic Partial Differential Equation; Partial Differential Equation.

Abbreviations—Alternating Direction Explicit (ADE); Alternating Direction Explicit Finite Difference Scheme (ADES); Alternating Direction Implicit (ADI); Alternating Direction Implicit Finite Difference Scheme (ADIS); Backward Difference Approximation (BDA); Central Difference Approximation (CDA); Finite Difference Approximation (FDA); Iterative Alternating Decomposition Explicit (IADE); Partial Differential Equation (PDE).

I. INTRODUCTION

Fluid flow equations are usually difficult to solve analytically, because either the flow is described by a non-linear Partial Differential Equation (PDE) that is complicated or the medium properties are heterogeneous. In such cases, numerical methods can be employed to obtain reliably approximate solution. The main reason for preferring numerical methods is that solutions will be obtained for many problems which are not susceptible to analytical treatment and are also suited for computer oriented numerical methods. Partial Differential Equations (PDEs) form the basis of very many mathematical models of physical, chemical and biological phenomena, and more recently they spread into economics, financial forecasting image processing and other fields. According to Morton & Mayers (2005) investigations of the predictions of PDE models of such phenomena, it is often necessary to approximate their solution numerically, commonly in combination with the analysis of simple special cases; while in some of the recent instances the numerical models play an almost independent role. Parabolic partial differential equations in one, two or three space dimensions with over-specified boundary data feature in the mathematical modeling of many important phenomena. Douglas & Peaceman (1955) found that while a significant body of knowledge about the theory and numerical methods for parabolic partial differential equations with classical boundary conditions has been accumulated, not much has been extended to parabolic partial differential equations with over-specified boundary data. According to Ming-shu & Tong-ke (2000) we often meet the problem of solving equation of parabolic type in many fields such as seepage, diffusion, heat conduction among others. The motivations of this research are:
• In compression to determine the rate of settlement of a foundation or structures
• Evaluation of strength and safety factors of an embankment
• Determination of rate of leakage through an earth dam

The general objective of the study was to solve the one-dimensional third order seepage parabolic partial differential equation given in equation (1.1) with the initial condition $u(x,0) = 0$ for $t = 0$ and $0 \leq x \leq \infty$, and boundary conditions

$$u(\infty, t) = g_1(t) = 0 \text{ for } x = \infty \text{ and } 0 \leq t \leq \infty$$

$$u(0, t) = g_2(t) = 1 - e^{-t} \text{ for } x = 0 \text{ and } 0 \leq t \leq \infty$$

(1.1)

numerically using finite difference method.

The specific objectives are:
• To formulate an algorithm for solving the seepage equation for fluid flow
• To discretize the system of differential equations governing fluid flow.
• To develop computer codes for implementing algorithm for solving fluid flow problems

The contributions of this research are:
• To enable scientists and engineers in understanding fluid flow and hence trigger ways of prevention of seepage through porous materials
• To apply knowledge of fluid flow in concrete structures of foundations and determine the rate of settlement of foundations.

The results of this study would be of great use to civil engineers, earthquake engineers and other engineers and would also contribute significantly to mathematical knowledge in this selected field of study.

II. Literature Review

Noye & Hayman (1994) used ADI to solve the two dimensional time-dependent heat equations subject to a constant coefficient. Dehghan (2002) used ADI methods for solving elliptic problems. Alias & Islam (2010) used Alternating Group Explicit (AGE) method and Iterative Alternating Decomposition Explicit (IADE) method to solve a two-dimensional and three-dimensional in PDE problems. Antar & Mokheimer (2011) used spreadsheets programs to solve a three dimensional equation for numerical solutions by using finite difference solutions which are the most appropriate. Taha & Ablowitz (1984) solved a generalized Korteweg-De Vries equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \quad u(x,0) = u_0(x)$$

(1.2)

which is closely similar to seepage equation using operator splitting techniques. Ronald & Lee (1988) solved

$$\frac{\partial u}{\partial t} = a \frac{\partial^2 u}{\partial x^2} + b \frac{\partial^3 u}{\partial x^3 \partial t}$$

(1.3)

where $a$ and $b$ are constants subject to some general initial conditions using Fourier transforms. The solution was in terms of integrals. Abdelfatah & Nabil (2006) solved a similar inhomogeneous problem with integral conditions and obtained non-numerical results.

Barenblatt et al., (1960) solved the generalized seepage equation,

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial^3 u}{\partial x^3} + f(x,t)$$

(1.4)

subject to the boundary conditions $u(0,t) = 1 - e^{-t}$ and $u(\infty, t) = 0$ and the initial condition $u(x,0) = 0$ with $\alpha = \beta = 1$ and $f(x,t) = 0$ analytically by using Laplace transforms and complex variable theory and merely obtained a non-closed form solution i.e.

$$u(x,t) = 1 - \frac{2}{\pi} \int_0^\infty \sin(va) \exp\left( - \frac{v^2 t}{1+v^2} \right) dv$$

(1.5)

The integral cannot be evaluated analytically. Moreover the integral is improper because the integrand tends to infinity and zero at the lower limit and upper limit respectively.

This is of no practical consequence. There is need therefore get numerical solutions which can be used to describe a physical phenomenon.

2.1. Partial Differential Equation

Many physical problems in science and engineering when modeled mathematically leads to a PDE. According to Rao Sankar (2004, 2004A), an equation which involves independent variables (usually denoted by $x, y, z, t, \cdots$), a dependent function $u$ of these variables and the partial derivatives of the dependent function $u$ with respect to the independent variable such as

$$F(x, y, z, \cdots, u, u_x, u_y, \cdots, u_{xx}, u_{yy}, u_{zz}, \cdots) = 0$$

is called a PDE. An example is the seepage equation $u_i = u_{xx} + u_{xt}$. The order of the PDE is the order of the highest partial derivative occurring in the equation. The degree of a PDE is the power of the order of the PDE. The seepage equation is of order three and first degree.

![Figure 1: Typical Inside Node of a Finite Difference Mesh](image-url)
Let the subscript \(i\) represent \(x\) co-ordinate and the subscript \(j\) to represent time. Let the mesh spacing in \(x\) and \(t\) directions be denoted by \(\Delta x\) and \(\Delta t\). Thus \(u(x, t) \approx U_{i,j}\). Let us assume that the function \(u\) and its partial derivatives are continuous. Then the Finite Difference Approximations (FDA) to derivatives can be obtained from Taylor’s series expansions of two variables as done by Jain (1984). The Taylor’s series expansion of \(U_{i+1,j}\) about the grid point \((i, j)\) gives

\[
U_{i+1,j} = U_{i,j} + \left[ \Delta t \frac{\partial U}{\partial t} + \frac{\Delta x^2}{2} \frac{\partial^2 U}{\partial x^2} + \frac{\Delta x^4}{6} \frac{\partial^4 U}{\partial x^4} + \ldots \right]_{i,j} \quad (2.1)
\]

Similarly,

\[
U_{i-1,j} = U_{i,j} - \left[ \Delta t \frac{\partial U}{\partial t} - \frac{\Delta x^2}{2} \frac{\partial^2 U}{\partial x^2} + \frac{\Delta x^4}{6} \frac{\partial^4 U}{\partial x^4} + \ldots \right]_{i,j} \quad (2.2)
\]

The Taylor’s series expansions of \(U_{i,j+1}\) and \(U_{i,j-1}\) will be

\[
U_{i,j+1} = U_{i,j} + \left[ \Delta t \frac{\partial U}{\partial t} + \frac{\Delta x^2}{2} \frac{\partial^2 U}{\partial x^2} + \frac{\Delta x^4}{6} \frac{\partial^4 U}{\partial x^4} + \ldots \right]_{i,j} \quad (2.3)
\]

\[
U_{i,j-1} = U_{i,j} - \left[ \Delta t \frac{\partial U}{\partial t} + \frac{\Delta x^2}{2} \frac{\partial^2 U}{\partial x^2} + \frac{\Delta x^4}{6} \frac{\partial^4 U}{\partial x^4} + \ldots \right]_{i,j} \quad (2.4)
\]

Solving for \(\frac{\partial U}{\partial x}\), (2.2) gives

\[
\left( \frac{\partial U}{\partial x} \right)_{i,j} = \frac{U_{i+1,j} - U_{i,j}}{\Delta x} - \frac{\Delta x^2}{2} \frac{\partial^2 U}{\partial x^2} + \ldots
\]

This can be written as

\[
\left( \frac{\partial U}{\partial x} \right)_{i,j} = \frac{U_{i+1,j} - U_{i,j}}{\Delta x} + O(\Delta x) \quad (2.5)
\]

while (3.3) gives

\[
\left( \frac{\partial U}{\partial x} \right)_{i,j} = \frac{U_{i+1,j} - U_{i,j-1}}{\Delta t} + O(\Delta t) \quad (2.6)
\]

In a similar manner (3.4) and (3.5) gives

\[
\left( \frac{\partial U}{\partial t} \right)_{i,j} = \frac{U_{i,j+1} - U_{i,j}}{\Delta t} + O(\Delta t) \quad (2.7)
\]

and

\[
\left( \frac{\partial U}{\partial t} \right)_{i,j} = \frac{U_{i,j} - U_{i,j-1}}{\Delta t} + O(\Delta t) \quad (2.8)
\]

respectively. Equations (2.5) and (2.7) are the Forward Difference Approximations (FDAp) to the derivatives \(\frac{\partial U}{\partial x}\) and \(\frac{\partial U}{\partial t}\) in that order, while (2.6) and (2.8) are their Backward Difference Approximations (BDA). All the Equations (2.5), (2.6), (2.7) and (2.8) are first order accurate. That is they have a truncation error of order \(\Delta x\) and \(\Delta t\).

Subtracting (2.2) from (2.1) and re-arranging further we get

\[
\frac{\partial U}{\partial t} + \frac{\partial^2 U}{\partial x^2} + \ldots
\]

This can be written as

\[
\frac{\partial U}{\partial t} = \frac{U_{i+1,j} - U_{i,j}}{2\Delta t} + O(\Delta x^2) \quad (2.9)
\]

Similarly for central difference, it can be shown that;

\[
\frac{\partial U}{\partial t} = \frac{U_{i,j+1} - U_{i,j-1}}{2\Delta t} + O(\Delta t^2) \quad (2.10)
\]

Equations (2.10) and (2.11) are Central Difference Approximation (CDA) to the derivatives \(\frac{\partial U}{\partial x}\) and \(\frac{\partial U}{\partial t}\), and have a truncation error of orders \(O(\Delta x^2)\) and \(O(\Delta t^2)\) respectively. The central difference approximation to a second order partial derivative \(\frac{\partial^2 U}{\partial x^2}\) can similarly be obtained by adding (3.2) and (3.3). Thus

\[
\frac{\partial^2 U}{\partial x^2} = \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{\Delta x^2} - \frac{(\Delta x)^2}{6} \frac{\partial^4 U}{\partial x^4} + \ldots
\]

which can be written as

\[
\frac{\partial^2 U}{\partial x^2} = \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{\Delta x^2} + O(\Delta x^4) \quad (2.9)
\]

which is second order accurate. The central difference approximation to a second order partial derivative \(\frac{\partial^2 U}{\partial t^2}\) can similarly be obtained by adding (2.3) and (2.4). Thus

\[
\frac{\partial^2 U}{\partial t^2} = \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{(\Delta t)^2} - \frac{(\Delta t)^2}{6} \frac{\partial^4 U}{\partial t^4} + \ldots
\]

which can be written as

\[
\frac{\partial^2 U}{\partial t^2} = \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{(\Delta t)^2} + O(\Delta t^4) \quad (2.12)
\]

which is second order accurate.

### 2.2. Discretization of Equation (1.1)

Discretization of Equation (1.1) could be obtained by replacing partial derivatives with their difference analogues. \(u_{xx}\) is approximated by Equation (2.11) and \(u_{tt}\) is approximated by Equation (2.12).

The expressions for mixed derivatives \(u_{xxt}\) can be obtained by differentiating with respect to each variable in turn. Thus an example of our derivative \(\frac{\partial^3 u}{\partial x^2 \partial t}\) will be

\[
\left( \frac{\partial^3 u}{\partial x^2 \partial t} \right)_{i,j} = \frac{\partial^2}{\partial x^2} \left( \frac{\partial u}{\partial t} \right)_{i,j} = \frac{\partial^2}{\partial x^2} \left( \frac{\partial u}{\partial t} \right)_{i-1,j} - \frac{\partial^2}{\partial x^2} \left( \frac{\partial u}{\partial t} \right)_{i+1,j}
\]

\[
= \frac{2\Delta x}{2\Delta t} \left( \frac{U_{i+1,j} - U_{i,j}}{2\Delta t} - \frac{U_{i,j} - U_{i-1,j}}{2\Delta t} \right)
\]

and

\[
\left( \frac{\partial^3 u}{\partial x^2 \partial t} \right)_{i,j} = \frac{\partial^2}{\partial x^2} \left( \frac{\partial u}{\partial t} \right)_{i,j+1} - \frac{\partial^2}{\partial x^2} \left( \frac{\partial u}{\partial t} \right)_{i,j-1}
\]

\[
= \frac{2\Delta x}{2\Delta t} \left( \frac{U_{i,j+1} - U_{i,j}}{2\Delta t} - \frac{U_{i,j} - U_{i,j-1}}{2\Delta t} \right)
\]
with the concepts of stability and convergence. In this study therefore, we would test these concepts for the numerical schemes developed. We would then solve the equation subject to the given boundary and initial value conditions. Mathematica software would be used to generate values in this study.

3.1.2. Discretization of the Partial Derivatives

The finite difference technique basically involves replacing the partial derivatives occurring in the partial differential equation as well as in the boundary and initial conditions by their corresponding finite difference approximations and then solving the resulting linear algebraic system of equations by a direct method or a standard iterative procedure. The numerical values of the dependent variable were obtained at the points of intersection of the parallel lines, called mesh points or nodal point.

3.1.3. Numerical Schemes and their Stability Analysis

We developed two numerical schemes and analyzed their stability as follows;

A. Alternating Direction Explicit Scheme (ADES)

From our equation $u_t = u_{xx} + u_{xt}$, we discretize it by replacing $u_t$ by forward difference in equation (2.10), $u_{xx}$ by central difference in equation (2.11) and $u_{xt}$ by the derived forward-central difference equation (2.13), we obtained:

$$\frac{u_{i,j+1} - u_{i,j}}{h} = \frac{u_{i+1,j+1} - 2u_{i+1,j} + u_{i+1,j-1}}{h^2}$$

$$+ \frac{1}{h} \left[ 2u_{i,j+1} + u_{i-1,j+1} - u_{i+1,j} + 2u_{i,j} - u_{i-1,j} \right]$$

(3.1)

Multiplying (3.1) by $k$ and letting $\alpha = k/h^2$ and $\phi = 1/h^2$, the scheme (3.1) becomes

$$(1 + 2\phi)U_{i,j+1} - \phi U_{i-1,j+1} - (\alpha - \phi)U_{i+1,j+1}$$

$$= (1 + 2\phi - 2\alpha)U_{i,j} + (\alpha - \phi)U_{i-1,j} + (\alpha - \phi)U_{i+1,j}$$

(3.2)

B. Stability Analysis of Alternating Direction Explicit Scheme (ADES)

We use the matrix method to analyze stability of the scheme (3.2)

Expanding this scheme by taking $j = 0, i = 1, 2, 3, \ldots (N - 2), (N - 1)$, we get the system of equations

$$(1 + 2\phi)U_{1,j+1} - \phi U_{0,j+1} - (\alpha - \phi)U_{2,j+1} = (1 + 2\phi - 2\alpha)U_{1,j} + (\alpha - \phi)U_{0,j} + (\alpha - \phi)U_{2,j}$$

and so on.
Writing the system in matrix form we get

\[
\begin{bmatrix}
U_{l+1,j} \\
U_{l+2,j} \\
\vdots \\
U_{N-1,j}
\end{bmatrix} = \begin{bmatrix}
\phi U_{l,j} - (\alpha - \phi) U_{l,j+1} \\
\phi U_{l,j} - (\alpha - \phi) U_{l,j+1} \\
\vdots \\
\phi U_{N-1,j} - (\alpha - \phi) U_{N-1,j+1}
\end{bmatrix}
\]

The system can be written compactly as

\[(I + 2\phi C_{N-1}) U_{j+1} = [I + (\phi - \alpha) A_{N-1}] U_j + \bar{b}
\]

Where \( I \) is an identity matrix of order \((N - 1) \times (N - 1)\), \( \bar{b} \) is a constant vector where

\[
A_{N-1} = \begin{bmatrix}
2 & -1 & \cdots & 0 & 0 \\
-1 & 2 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & -1 & 2 & -1 \\
0 & \cdots & -1 & 2 & 0
\end{bmatrix}, \quad C_{N-1} = \begin{bmatrix}
1 & 1 & \frac{1}{2} & \cdots & 0 & 0 \\
1 & 1 & \frac{1}{2} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & -\frac{1}{2} & 1 & \frac{1}{2} & 1 \\
0 & 0 & -\frac{1}{2} & 1 & \frac{1}{2} & 1
\end{bmatrix}
\]

And \( \bar{b} = \begin{bmatrix}
\phi U_{0,j+1} - (\alpha - \phi) U_{0,j} \\
\vdots \\
\phi U_{N-1,j+1} - (\alpha - \phi) U_{N-1,j}
\end{bmatrix}\)

Therefore \( U_{j+1} = [I + (\phi - \alpha) A_{N-1}]^{-1} \] and \( \bar{b} = [I + (\phi - \alpha) A_{N-1}]^{-1} \)

where \( B = [I + (\phi - \alpha) A_{N-1}]^{-1}\)

\[
B = \begin{bmatrix}
I + (\phi - \alpha) A_{N-1}
\end{bmatrix}
\]

For a tridiagonal matrix, the modulus of the eigenvalues of the amplification matrix \( B \) should be less than or equal to unity.

\[
\left| \frac{1 + (\phi - \alpha) A_{N-1}}{1 + 2\phi} \right| \leq 1
\]

For \( \phi > 0 \), \( \alpha > 0 \) which is the stability criterion condition.

C. Alternating Direction Implicit Scheme (ADIS)

In this scheme, we replace \( u_i \) by the central difference approximation, \( u_{x+1} \) by the average of the \( j \)th level and the \((j+1)\)th level central differences and \( u_{x+2} \) by the derived Forward-central difference in our Equation (2.14). We obtain

\[
U_{j+1} - 2U_j + U_{j-1} + \frac{k^2}{h^2} \left( U_{j+1,1} + 2U_{j+1} + U_{j-1,1} \right)
\]

Expanding this scheme by taking \( \alpha = k/h^2 \) and \( \phi = 1/h^2 \), and re-arranging, the scheme becomes

\[
(1 + 2\phi + 2\alpha)U_{j+1} - (\alpha + \phi)U_{j-1,1} + (\alpha + \phi)U_{j+1,1} = (1 + 2\phi)U_{j+1} - \phi U_{j+1} - \phi U_{j+1}
\]

\[
(1 + 2\phi)U_{j+1} - (\alpha + \phi)U_{j-1,1} + (\alpha + \phi)U_{j+1,1} + (1 + 2\phi)U_{j+1} - \phi U_{j} - \phi U_{j}
\]

\[
= (1 + 2\phi)U_{j+1} - \phi U_{j} - \phi U_{j}
\]

\[
= (1 + 2\phi)U_{j+1} - \phi U_{j} - \phi U_{j}
\]

\[
= (1 + 2\phi)U_{j+1} - \phi U_{j} - \phi U_{j}
\]

D. Stability Analysis Alternating Direction Implicit Scheme (ADIS)

We use also the matrix method to analyze stability of the scheme (3.4)

Expanding this scheme by taking \( i = 1,2,3, \ldots, (N-2), (N-1) \), we get the system of equations

\[
(1 + 2\phi + 2\alpha)u_{i+1} - (\alpha + \phi)u_{i-1} = (1 + 2\phi)u_{i+1} - \phi u_{i+2} - \phi u_{i-2}
\]

\[
(1 + 2\phi)u_{i+1} - (\alpha + \phi)u_{i-1} + (\alpha + \phi)u_{i+1} + (1 + 2\phi)u_{i+1} - \phi u_{i} - \phi u_{i}
\]

\[
= (1 + 2\phi)u_{i+1} - \phi u_{i} - \phi u_{i}
\]

\[
= (1 + 2\phi)u_{i+1} - \phi u_{i} - \phi u_{i}
\]

\[
= (1 + 2\phi)u_{i+1} - \phi u_{i} - \phi u_{i}
\]

\[
= (1 + 2\phi)u_{i+1} - \phi u_{i} - \phi u_{i}
\]

\[
= (1 + 2\phi)u_{i+1} - \phi u_{i} - \phi u_{i}
\]

Therefore eigenvalues of amplification matrix
Writing the system in matrix form we get
\[
\begin{bmatrix}
1 + 2\alpha + 2\phi & -(\alpha + \phi) & \cdots & 0 & 0 \\
-(\alpha + \phi) & (1 + 2\alpha + 2\phi) & \cdots & -(\alpha + \phi) & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & (1 + 2\alpha + 2\phi) & -(\alpha + \phi) \\
0 & 0 & \cdots & -(\alpha + \phi) & (1 + 2\alpha + 2\phi)
\end{bmatrix}
\begin{bmatrix}
U_{1,j+1} \\
U_{2,j+1} \\
\vdots \\
U_{N-1,j+1} \\
U_{N,j+1}
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
\lambda \\
\lambda \\
\vdots \\
\lambda \\
\lambda
\end{bmatrix}
\]

The system can be written compactly as
\[
\left[I + (\alpha + \phi)G_{N-1} + (\alpha + \phi)B_{N-1}\right]U_{j+1} = \left[I + \phi C_{N-1}\right]U_j + \bar{e}
\]

where \(I\) is an identity matrix of order \((N-1) \times (N-1)\), \(\bar{e}\) is a constant vector and
\[
\begin{align*}
G_{N-1} & = \begin{bmatrix}
0 & -1 & \cdots & 0 \\
-1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0
\end{bmatrix}, \\
C_{N-1} & = \begin{bmatrix}
2 & 1 & \cdots & 0 \\
1 & 2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0
\end{bmatrix}, \\
B_{N-1} & = \begin{bmatrix}
0 & -1 & \cdots & 0 \\
-1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0
\end{bmatrix}
\end{align*}
\]

Therefore
\[
U_{j+1} = \left[I + (\alpha + \phi)G_{N-1} + (\alpha + \phi)B_{N-1}\right]^{-1}\left[I + \phi C_{N-1}\right]U_j + \frac{\phi}{\Delta t}F
\]

where, \(\Delta t = \frac{h}{k}\) and taking \(\phi = \frac{1}{\Delta t^2}\)

Eigenvectors of
\[
\begin{align*}
G_{N-1} & = 0 + 2\cos\left(\frac{m\pi}{16}\right) - 2\sin^2\left(\frac{m\pi}{32}\right) \\
C_{N-1} & = 2 + 2\cos\left(\frac{m\pi}{16}\right) = 4 - 4\sin^2\left(\frac{m\pi}{32}\right) \\
B_{N-1} & = 0 + 2\sqrt{1}\cos\left(\frac{m\pi}{16}\right) = 2\cos\left(\frac{m\pi}{16}\right) = 2 - 4\sin^2\left(\frac{m\pi}{32}\right)
\end{align*}
\]

Eigenvectors of
\[
H = \left[I + \phi C_{N-1}\right]^{-1}\left[I + (\alpha + \phi)G_{N-1} + (\alpha + \phi)B_{N-1}\right] \leq 1
\]

Which meets the stability criterion condition for \(\phi < 0\), \(\alpha > 0\).

IV. RESULTS AND DISCUSSION

4.1. Case 1

If \(h = k = \frac{1}{2}\) and taking \(\alpha = \frac{\Delta t}{(\Delta t)^2}\)

4.1.1. Alternating Direction Explicit Scheme (ADES) Case 1

From the ADES (3.2), we fix \(j=0\) and vary \(i=1, 2, \ldots, 15\), as follows
\[
i=1: 9U_{1,i+1} + 4U_{2,i+1} + 4U_{3,i+1} = 5U_{1,i} - 2U_{2,i} - 2U_{3,i}
\]
\[
i=2: 9U_{2,i+1} + 4U_{3,i+1} = 5U_{1,i} - 2U_{2,i} - 2U_{3,i}
\]
\[
i=3: 9U_{3,i+1} = 5U_{1,i} - 2U_{2,i} - 2U_{3,i}
\]
\[
i=4: 9U_{1,i} - 4U_{2,i} - 4U_{3,i} = 5U_{1,i} - 2U_{2,i} - 2U_{3,i}
\]
\[
i=5: 9U_{1,i} - 4U_{2,i} - 4U_{3,i} = 5U_{1,i} - 2U_{2,i} - 2U_{3,i}
\]
\[
i=6: 9U_{1,i} - 4U_{2,i} - 4U_{3,i} = 5U_{1,i} - 2U_{2,i} - 2U_{3,i}
\]
\[
i=7: 9U_{1,i} - 4U_{2,i} - 4U_{3,i} = 5U_{1,i} - 2U_{2,i} - 2U_{3,i}
\]
\[
i=8: 9U_{1,i} - 4U_{2,i} - 4U_{3,i} = 5U_{1,i} - 2U_{2,i} - 2U_{3,i}
\]
\[
i=9: 9U_{1,i} - 4U_{2,i} - 4U_{3,i} = 5U_{1,i} - 2U_{2,i} - 2U_{3,i}
\]
\[
i=10: 9U_{1,i} - 4U_{2,i} - 4U_{3,i} = 5U_{1,i} - 2U_{2,i} - 2U_{3,i}
\]
\[
i=11: 9U_{1,i} - 4U_{2,i} - 4U_{3,i} = 5U_{1,i} - 2U_{2,i} - 2U_{3,i}
\]
\[
i=12: 9U_{1,i} - 4U_{2,i} - 4U_{3,i} = 5U_{1,i} - 2U_{2,i} - 2U_{3,i}
\]
\[
i=13: 9U_{1,i} = 5U_{1,i} - 2U_{2,i} - 2U_{3,i}
\]
\[
i=14: 9U_{1,i} - 4U_{2,i} - 4U_{3,i} = 5U_{1,i} - 2U_{2,i} - 2U_{3,i}
\]
\[
i=15: 9U_{1,i} - 4U_{2,i} - 4U_{3,i} = 5U_{1,i} - 2U_{2,i} - 2U_{3,i}
\]

In matrix form we got:
\[
\begin{bmatrix}
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}
\]

Using Mathematica Software we got following solutions:
\[
U_{1,s} = 0.3853481, U_{2,s} = 0.2349126, U_{3,s} = 0.1432053, U_{4,s} = 0.0872992, U_{5,s} = 0.05321801, \\
U_{6,s} = 0.0324413, U_{7,s} = 0.01977492, U_{8,s} = 0.01250272, U_{9,s} = 0.00734285, \\
U_{10,s} = 0.00468771, U_{11,s} = 0.00271205, U_{12,s} = 0.00163341, \\
U_{13,s} = 0.0009629675, U_{14,s} = 0.0005333385, U_{15,s} = 0.0002370381
\]

4.2. Case 2

ADIS and \(j=3\) : The seepage flow decreases with increase in \(i=1, 2, \ldots, 15\) but more slowly than in the same scheme in case 1 and also decreases more slowly than in the corresponding case 2 ADE scheme.
This is due to large volume of water at the surface and also at the source. The fluid flow also experiences friction due to viscosity and medium particles such as soil.

![Graphical Presentation of ADES Case I](image1)

**Figure 2: Graphical Presentation of ADES Case I**

We note that for each i value, i=1,2,………,15, the seepage flow increased with increase in j value and for each j value, j=0,1,2,3, the seepage flow decreases as i-value increases steadily.

![Graphical Presentation for ADES Results](image2)

**Figure 3: Graphical Presentation for ADIS Results with h = k = 1/2**

We note that for each i value, i=1,2,………,15, the seepage flow increased with increase in j value and the increase is more than that of the corresponding ADES case. For each j value, j=0,1,2,3, the seepage flow decreases as i-value increases steadily but at a slower pace than in the corresponding ADES case.

![Graphical Presentation with ADES](image3)

**Figure 4: Graphical Presentation for ADES Results with h = k = 1/4**

We note that for each i value, i=1,2,………,15, the seepage flow increased with increase in j value and for each j value, j=0,1,2,3, the seepage flow decreases as i-value increases steadily.

<table>
<thead>
<tr>
<th>J=0</th>
<th>J=1</th>
<th>J=2</th>
<th>J=3</th>
</tr>
</thead>
<tbody>
<tr>
<td>i=1</td>
<td>0.3853481</td>
<td>0.5601236</td>
<td>0.6670508</td>
</tr>
<tr>
<td>i=2</td>
<td>0.2349126</td>
<td>0.3694944</td>
<td>0.4678016</td>
</tr>
<tr>
<td>i=3</td>
<td>0.1432053</td>
<td>0.2428888</td>
<td>0.3245777</td>
</tr>
<tr>
<td>i=4</td>
<td>0.08729923</td>
<td>0.1586531</td>
<td>0.2231869</td>
</tr>
<tr>
<td>i=5</td>
<td>0.05321801</td>
<td>0.1031687</td>
<td>0.1523049</td>
</tr>
<tr>
<td>i=6</td>
<td>0.03244131</td>
<td>0.0682435</td>
<td>0.1039865</td>
</tr>
<tr>
<td>i=7</td>
<td>0.01977492</td>
<td>0.04313089</td>
<td>0.06951022</td>
</tr>
<tr>
<td>i=8</td>
<td>0.01025227</td>
<td>0.02774829</td>
<td>0.04649375</td>
</tr>
<tr>
<td>i=9</td>
<td>0.007342285</td>
<td>0.01779631</td>
<td>0.03087895</td>
</tr>
<tr>
<td>i=10</td>
<td>0.004468771</td>
<td>0.0111363</td>
<td>0.02018054</td>
</tr>
<tr>
<td>i=11</td>
<td>0.00271205</td>
<td>0.00721911</td>
<td>0.0131101</td>
</tr>
<tr>
<td>i=12</td>
<td>0.001633341</td>
<td>0.004524799</td>
<td>0.00813458</td>
</tr>
<tr>
<td>i=13</td>
<td>0.0009629675</td>
<td>0.002764718</td>
<td>0.005125716</td>
</tr>
<tr>
<td>i=14</td>
<td>0.0005333358</td>
<td>0.0018757447</td>
<td>0.002992201</td>
</tr>
<tr>
<td>i=15</td>
<td>0.0002370381</td>
<td>0.0007133674</td>
<td>0.001364700</td>
</tr>
</tbody>
</table>

**Table 1: Case 1 ADIS Results**

<table>
<thead>
<tr>
<th>J=0</th>
<th>J=1</th>
<th>J=2</th>
<th>J=3</th>
</tr>
</thead>
<tbody>
<tr>
<td>i=1</td>
<td>0.4214125</td>
<td>0.6045336</td>
<td>0.6939972</td>
</tr>
<tr>
<td>i=2</td>
<td>0.2803938</td>
<td>0.4217462</td>
<td>0.5042542</td>
</tr>
<tr>
<td>i=3</td>
<td>0.1872905</td>
<td>0.2936422</td>
<td>0.3647182</td>
</tr>
<tr>
<td>i=4</td>
<td>0.1248562</td>
<td>0.2040735</td>
<td>0.267184</td>
</tr>
<tr>
<td>i=5</td>
<td>0.0832128</td>
<td>0.1438805</td>
<td>0.188541</td>
</tr>
<tr>
<td>i=6</td>
<td>0.05547825</td>
<td>0.0980604</td>
<td>0.1348399</td>
</tr>
<tr>
<td>i=7</td>
<td>0.03697159</td>
<td>0.0678015</td>
<td>0.09610735</td>
</tr>
<tr>
<td>i=8</td>
<td>0.02462878</td>
<td>0.04678452</td>
<td>0.06825468</td>
</tr>
<tr>
<td>i=9</td>
<td>0.01638662</td>
<td>0.03219675</td>
<td>0.04826654</td>
</tr>
<tr>
<td>i=10</td>
<td>0.01087748</td>
<td>0.02206475</td>
<td>0.03392722</td>
</tr>
<tr>
<td>i=11</td>
<td>0.00718124</td>
<td>0.0150059</td>
<td>0.02362218</td>
</tr>
<tr>
<td>i=12</td>
<td>0.004861886</td>
<td>0.001004907</td>
<td>0.01615454</td>
</tr>
<tr>
<td>i=13</td>
<td>0.002962843</td>
<td>0.006509991</td>
<td>0.01064674</td>
</tr>
<tr>
<td>i=14</td>
<td>0.0001773676</td>
<td>0.003884804</td>
<td>0.006444104</td>
</tr>
<tr>
<td>i=15</td>
<td>0.00008019726</td>
<td>0.00181355</td>
<td>0.003034412</td>
</tr>
</tbody>
</table>

**Table 2: Case 1 ADIS Results**
But the seepage flow decrease is slower than the corresponding ADES scheme in case 1 above.

<table>
<thead>
<tr>
<th>Case 2 ADIS Results</th>
<th>J=0</th>
<th>J=1</th>
<th>J=2</th>
<th>J=3</th>
</tr>
</thead>
<tbody>
<tr>
<td>i=1</td>
<td>0.5054709</td>
<td>0.7025233</td>
<td>0.7845406</td>
<td>0.8229078</td>
</tr>
<tr>
<td>i=2</td>
<td>0.4040949</td>
<td>0.5704533</td>
<td>0.6470262</td>
<td>0.6885759</td>
</tr>
<tr>
<td>i=3</td>
<td>0.3229234</td>
<td>0.462865</td>
<td>0.5329258</td>
<td>0.5750527</td>
</tr>
<tr>
<td>i=4</td>
<td>0.2578982</td>
<td>0.3751907</td>
<td>0.4382598</td>
<td>0.4791834</td>
</tr>
<tr>
<td>i=5</td>
<td>0.2057679</td>
<td>0.303697</td>
<td>0.3596916</td>
<td>0.3982389</td>
</tr>
<tr>
<td>i=6</td>
<td>0.1639260</td>
<td>0.2453305</td>
<td>0.2944249</td>
<td>0.3298622</td>
</tr>
<tr>
<td>i=7</td>
<td>0.1302804</td>
<td>0.1975912</td>
<td>0.2401146</td>
<td>0.2720229</td>
</tr>
<tr>
<td>i=8</td>
<td>0.1031488</td>
<td>0.1584287</td>
<td>0.1947920</td>
<td>0.2229692</td>
</tr>
<tr>
<td>i=9</td>
<td>0.0811747</td>
<td>0.1261562</td>
<td>0.1567996</td>
<td>0.1811884</td>
</tr>
<tr>
<td>i=10</td>
<td>0.06325932</td>
<td>0.09937964</td>
<td>0.1247361</td>
<td>0.1453703</td>
</tr>
<tr>
<td>i=11</td>
<td>0.04850689</td>
<td>0.07693952</td>
<td>0.09740956</td>
<td>0.1143734</td>
</tr>
<tr>
<td>i=12</td>
<td>0.0361798</td>
<td>0.0578613</td>
<td>0.07379606</td>
<td>0.08719519</td>
</tr>
<tr>
<td>i=13</td>
<td>0.02566171</td>
<td>0.04131437</td>
<td>0.0530043</td>
<td>0.06294429</td>
</tr>
<tr>
<td>i=14</td>
<td>0.0164267</td>
<td>0.02657653</td>
<td>0.03424338</td>
<td>0.04081506</td>
</tr>
<tr>
<td>i=15</td>
<td>0.008013024</td>
<td>0.01300325</td>
<td>0.01679745</td>
<td>0.0200664</td>
</tr>
</tbody>
</table>

Figure 5: Graphical Presentation for ADIS Results with \( h = k = 1/4 \)

We note that for each i-value, \( i=1,2,\ldots,15 \), the seepage flow increased with increase in j value and for each j-value, \( j=0,1,2,3 \), the seepage flow decreases as i-value increases steadily

But the seepage flow decrease is slower than the corresponding ADES scheme in case 1 above also the flow is slower than the ADES scheme in case 2.

4.3. Assumptions of the Study

The following assumptions were made:
- The fluid flow was assumed to be laminar.
- The fluid particles were assumed to be irrotational.
- The fluid flow was steady, perfect and incompressible.
- The variation of density of the fluid flow is negligible.

V. Conclusions

The study managed to analyze Finite Difference method for parabolic partial differential equations, results were obtained, and tabulated and graphical presentations made. The two schemes developed namely Alternating Direction Explicit and Alternating Direction Implicit schemes had added mesh points, that is \( (U_{m+2,n}), (U_{m+2,n+1}), (U_{m-2,n}), (U_{m-2,n+1}) \).

as seen in Equation (3.14) led to mesh refinement. The schemes developed were stable for all values \( \phi > 0 \) and \( \alpha > 0 \). We found that the smaller the mesh sizes, the more finely the results.

From the diagrams, the following can be interpreted:
- The surface of the plot is not smooth because the differential equation is satisfied only at a selected number of discrete nodes within the region of integration.
- For a given value of x, \( u(x,t) \) increases to nearly one as t tends to infinity.
- For a given value of t, \( u(x,t) \) decreases to nearly zero as x tends to infinity.
- The smaller the mesh size the more slowly the seepage tends to zero.

The solutions \( u(x,t) \) of equation (1.1) decrease with increase length of the fluid seepage from the source. Seepage is a very slow process. Fluid seeps few meters in several years. That is why the slope of the surface decreases slowly.

The solutions show that as \( x \to \infty \), the fluid flow decreases and it may eventually cease.

Acknowledgement

Am so much grateful to my academic committee members Prof. J.K. Sigey, Dr.J. Okelo, Dr.J. Okwoyo for their incredible support in all the time they took working on my research. I am also expressing my profound gratitude to Jomo Kenyatta University of Agriculture and Technology (JUAT) for offering me a chance to study.

References

Affiliation: Jomo Kenyatta University of Agriculture and Technology, (JKUAT), Kenya.

Dr. Okelo Jeconia Abonyo holds a PhD in Applied Mathematics from Jomo Kenyatta University of Agriculture and Technology as well as a Master of science degree in Mathematics and first class honors in Bachelor of Education, Science; specialized in Mathematics with option in Physics, both from Kenyatta University. I have dependable background in Applied Mathematics in particular fluid dynamics, analyzing the interaction between velocity field, electric field and magnetic field. Has a hand on experience in implementation of curriculum at secondary and university level. He has demonstrated sound leadership skills and has the ability to work on new initiatives as well as facilitating teams to achieve set objectives. He has good analytical, design and problem solving skills.

Affiliation: Jomo Kenyatta University of Agriculture and Technology, (JKUAT), Kenya.

Dr. Okwoyo James Marita holds a Bachelor of Education degree in Mathematics and Physics from Moi University, Kenya, Master Science degree in Applied Mathematics from the University of Nairobi and PhD in applied mathematics from Jomo Kenyatta University of Agriculture and Technology, Kenya.

Affiliation: University of Nairobi, Chiromo Campus School of Mathematics P.O. 30197-00100 Nairobi, Kenya.

He is currently a lecturer at the University of Nairobi (November 2011 – Present) responsible for carrying out teaching and research duties. He plays a key role in the implementation of University research projects and involved in its publication. He was an assistant lecturer at the University of Nairobi (January 2009 – November 2011). He has published 7 papers on heat transfer in respected journals.

Supervision of postgraduate students

- Masters of science in applied mathematics: (8 completed and 8 ongoing)

Affiliation: Jomo Kenyatta University of Agriculture and Technology, (JKUAT), Kenya.

Prof. Johana Kibet Sigei holds a Bachelor of Science degree in mathematics and computer science First Class honors from Jomo Kenyatta University of Agriculture and Technology, Kenya, Master of Science degree in Applied Mathematics from Kenyatta University and a PhD in applied mathematics from Jomo Kenyatta University of Agriculture and Technology, Kenya.

Affiliation: Jomo Kenyatta University of Agriculture and Technology, (JKUAT), Kenya.

He is currently the Director, JKuat, Kisii CBD. He has been the substantive chairman - Department of Pure and Applied mathematics –JKuat (January 2007 to July- 2012). He holds the rank of Associate Professor in Applied Mathematics in Pure and Applied Mathematics department – JKut since November 2009 to date. He has published 9 papers on heat transfer, MHD and Traffic models in respected journals.

Teaching experience: 2000 to date- postgraduate programmes:

(JKUAT)

- Master of science in applied mathematics

Affiliation: Jomo Kenyatta University of Agriculture and Technology, (JKUAT), Kenya.

Nelson MLS Nyachwaya Nelson was born on 22nd December, 1956 in Keroka town, Nyamira county, Kenya. He holds a Bachelor of Science degree, and specialized in Mathematics, from University of Nairobi, Kenya and is currently pursuing a Master of Science degree in Applied Mathematics from Jomo Kenyatta University of Agriculture and Technology, Kenya.

Affiliation: Jomo Kenyatta University of Agriculture and Technology, (JKUAT), Kenya.

He holds a Bachelor of Technology degree in Mechanical Engineering from Moi University, Kenya. Master of Science degree in Applied Mathematics from University of Nairobi, senior lecturer in Pure and Applied Mathematics.

Affiliation: Jomo Kenyatta University of Agriculture and Technology, (JKUAT), Kenya.

He has 28 years of teaching experience. He has published 15 papers in respected journals.

Supervision of postgraduate students

- masters of science in applied mathematics: (13 completed, 8 ongoing)

Affiliation: Jomo Kenyatta University of Agriculture and Technology, (JKUAT), Kenya.

Masters of science degree in Applied Mathematics from Moi University, Kenya.

Affiliation: Jomo Kenyatta University of Agriculture and Technology, (JKUAT), Kenya.

He has 25 years of teaching experience. He has published 23 papers in respected journals.

Supervision of postgraduate students

- Doctor of philosophy: thesis (3 completed, 5 ongoing)

Affiliation: Jomo Kenyatta University of Agriculture and Technology, (JKUAT), Kenya.

He has 15 years of teaching experience. He has published 17 papers in respected journals.

Supervision of postgraduate students

- Doctor of philosophy: thesis (3 completed, 5 ongoing)

Affiliation: Jomo Kenyatta University of Agriculture and Technology, (JKUAT), Kenya.

He has 20 years of teaching experience. He has published 20 papers in respected journals.

Supervision of postgraduate students

- Doctor of philosophy: thesis (3 completed, 5 ongoing)

Affiliation: Jomo Kenyatta University of Agriculture and Technology, (JKUAT), Kenya.

He has 23 years of teaching experience. He has published 24 papers in respected journals.

Supervision of postgraduate students

- Doctor of philosophy: thesis (3 completed, 5 ongoing)

Affiliation: Jomo Kenyatta University of Agriculture and Technology, (JKUAT), Kenya.

He has 26 years of teaching experience. He has published 26 papers in respected journals.

Supervision of postgraduate students

- Doctor of philosophy: thesis (3 completed, 5 ongoing)