Mathematical Analysis of Elastically Supported Beams with Application to Excited Bridge Vibrations

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***Abstract*—**Dynamic structures operate under hostile environmental conditions and with minimum maintenance. Vibration levels are expected to be low for the smooth and quiet running of these structures. It is therefore essential that the effects and control of the vibrations of these structures are clearly understood so that effective analysis, design and modification may be carried out. The numerical solutions of excited vibrations of the Bernoulli-Euler type beam in the general case of external loading function were analyzed. The vertical deflection of a simply supported and clamped beam considered under a uniform load using the Finite Difference Method. The governing differential equation is pre-described by the Bernoulli-Euler beam equation which is a fourth order differential equation. A model of the Bernoulli-Euler beam equation that governs excited bridge vibrations was solved using Finite Difference Method (FDM). This involves Discretization of Bernoulli-Euler beam equation employing Crank Nicholson scheme after approximation. The solutions are used to investigate the effect of cross sectional area and external forcing term on transverse displacement of a beam on excited bridges. The effects of these parameters are discussed and also presented graphically. It was found that an increase in the forcing term leads to an increase in transverse displacement of a beam while an increase in cross sectional area results to a decrease in transverse displacement of a beam.

***Keywords*—**Beam; Excited Bridge; Parabolic Partial Differential Equation; Partial Differential Equation; Vibrations

***Abbreviations*—**Backward Difference Approximation (BDA); Central Difference Approximation (CDA); Central Difference Scheme (CDS); Finite Difference Approximation (FDA); Iterative Alternating Decomposition Explicit (IADE); Partial Differential Equation (PDE).

I. Introduction and Literature Review

IN this section, introduction of the area of study is done, literature review, geometry of problem, statement of problem, objectives and significance of study are looked at.

1.1. Introduction

A beam is a structural element that is capable of withstanding load primarily by resisting against bending. The bending force induced into the material of the beam as a result of the external loads, own weight, span and external reactions to these loads is called a bending moment. Beams are characterized by their profile (shape of cross-section), their length, and their material. Historically beams were squared timbers but are also metal, stone, or combinations of wood and metal such as a flitch beam. Beams generally carry vertical gravitational forces but can also be used to carry horizontal loads (e.g., loads due to an earthquake or wind or in tension to resist rafter thrust as a tie beam or (usually) compression as a collar beam). The loads carried by a beam are transferred to columns, walls, or girders, which then transfer the force to adjacent structural compression members. Compression members are structural elements that are pushed together or carry a load; more technically they are subjected only to axial compressive forces. That is, the loads are applied on the longitudinal axis through the centroid of the member cross section, and the load over the cross sectional area gives the stress on the compressed member. In buildings posts and columns are almost always compression members as are the top chord of trusses. Beams are traditionally descriptions of building or civil engineering structural elements, but smaller structures such as truck or automobile frames, machine frames, and other mechanical or structural systems contain beam structures that are designed and analyzed in a similar fashion.

Beams are one of the components used in structural engineering. Beams can be in 1-dimension, 2-dimension or 3-dimension. They can be horizontal, vertical and also at angles. We shall analyze a uniform elastic beam, simply supported length L and subjected to vertical forces acting in the principal plane of a symmetrical cross-section.

The design of such beams can be complex but is essentially intended to ensure that the beam can safely carry the load it is intended to support. This will include its own self-weight, the weight of the structure it is supporting and what is often referred to as “live load” being the weight of people and furnishings in buildings or the weight of road or rail traffic in bridges. In addition to the requirements for the beam to safely carry the intended design loads there are other factors that have to be considered including assessing the likely deflection of the beam under load. If beams deflect excessively then this can cause visual distress to the users of the building and can lead to damage of parts of the building including brittle partition dividers between rooms and services such as water and heating pipes and ductwork.

Beam equations have historical importance, as they have been the focus of attention for prominent scientists such as Leonardo da Vinci (14^{th} Century) and Daniel Bernoulli (18^{th} Century). Practical applications of the beam equations are evident in mechanical structures built under the premise of beam theory. The importance of beam theory has been outlined in the literature over the years. Examples include the construction of high-rise buildings, bridges across the rivers, air craft and heavy motor vehicles. In these structures, beams are used as the basis of supporting structures or as the main-frame foundation in axles. Without a proper knowledge of beam theory, the successful manufacture of such structures would be unfeasible and unsafe. The Euler-Bernoulli beam theory, sometimes called the classical beam theory, is the most commonly used. It is simple and provides reasonable engineering approximations for many problems.

1.2. Damping, Resonance and Cancellation

Damping is the restraining of vibratory motion, such as mechanical oscillations, noise, and alternating electric currents, by dissipation of energy. Shock absorbers in automobiles and carpet pads are examples of damping devices. Critical damping prevents vibration or is just sufficient to allow the object to return to its rest position in the shortest period of time. The automobile shock absorber is an example of a critically damped device. Additional damping causes the system to be over damped, which may be desirable, as in some door closers. The vibrations of an under damped system gradually taper off to zero.

Resonance is when a vibrating system or external force drives another system to oscillate with greater amplitude at a specific preferential frequency. Increase of amplitude as damping decreases and frequency approaches resonant frequency of a driven damped simple harmonic oscillator. Frequencies at which the response amplitude is a relative maximum are known as the system's resonant frequencies or resonance frequencies. At resonant frequencies, small periodic driving forces have the ability to produce large amplitude oscillations. This is because the system stores vibrational energy.

Resonance occurs when a system is able to store and easily transfer energy between two or more different storage modes (such as kinetic energy and potential energy in the case of a pendulum). However, there are some losses from cycle to cycle, called damping. When damping is small, the resonant frequency is approximately equal to the natural frequency of the system, which is a frequency of unforced vibrations. Some systems have multiple, distinct, resonant frequencies. Resonance phenomena occur with all types of vibrations or waves:

Mechanical resonance is the tendency of a mechanical system to absorb more energy when the frequency of its oscillations matches the system's natural frequency of vibration than it does at other frequencies. It may cause violent swaying motions and even catastrophic failure in improperly constructed structures including bridges, buildings, trains, and aircraft. When designing objects, engineers must ensure the mechanical resonance frequencies of the component parts do not match driving vibrational frequencies of motors or other oscillating parts, a phenomenon known as resonance disaster.

Avoiding resonance disasters is a major concern in every building, tower, and bridge construction project. As a countermeasure, shock mounts can be installed to absorb resonant frequencies and thus dissipate the absorbed energy.

When a single load crosses a beam there are particular ratios of the beam natural period and the time spent by the load crossing the structure that, in absence of damping, result in the total extinction of the beam free transverse oscillations associated to a particular mode. This is known as the cancellation phenomenon. Therefore both resonance and cancellation situations are related to the free vibration response of beams but resonance needs a repetitive loading pattern to develop while cancellation takes place with the passage of every single load.

1.3. Excited Vibrations

When analyzing the excited vibrations of realistic beam (with internal damping), the general solution of the homogeneous differential equation is a function which relatively fast tending to zero with respect to time due to internal and external damping. The full solution of the problem of excited vibrations, understood as the special solution of the non-homogeneous Equation (1),

$$\frac{\partial^2 u}{\partial t^2} + c^2 \frac{\partial^4 u}{\partial x^4} = f(x,t) \qquad (1)$$

has for excitation function of harmonic in time domain type, the form of sum of two components: connected with a set of natural frequencies, and connected with external loading frequency. Due to the same result of action of internal and external damping (as for natural vibrations), the components connected with set of natural frequencies (free vibrations) are functions relatively fast tending to zero with respect to time. Therefore, the solution of exited vibrations is usually those, connected with the only external loading frequency. Such case of vibrations is called a steady-state case. The complete solution of the problem which includes free vibrations and excited vibrations in general formulation, is called as transient vibrations.

1.4. Geometry of the Problem

Under gravity loads, the original length of the beam is slightly reduced to enclose a smaller radius arc at the top of the beam, resulting in compression, while the same original beam length at the bottom of the beam is slightly stretched to enclose a larger radius arc, and so is under tension. We shall analyze a uniform elastic beam, simply supported length L and subjected to vertical forces acting in the principal plane of a symmetrical cross-section as shown below.

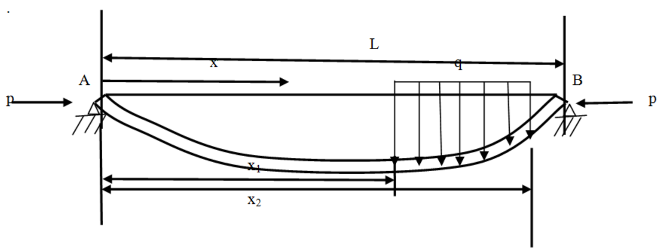

Figure 1: A Uniform Elastic Beam

q – Centre concentrated load.

L – The length AB.

$u(x,t)$– Deflection at (x,y). Where x is a one-dimension spatial variable point at time t.

P-End supports

1.5. Literature Review

Since the beginnings of the High-Speed era, several scientists have investigated the resonance phenomena in railway bridges. when a single load crosses a beam there are particular ratios of the beam natural period and the time spent by the load crossing the structure that, in absence of damping, result in the total extinction of the beam free transverse oscillations associated to a particular mode. This is known as the cancellation phenomenon and has been analyzed by a few authors. Qing & Elias (2016), in their article have examined the dynamic response of vehicles moving at high speeds and the sustaining horizontally curved(simple, continuous or multi-unit) bridges, when each subsystem is set into resonance. They observed the following, first, the impact factors show the same pattern for the vertical and radial directions, and second, the suspension damping can alleviate the resonance response of the car body even when the vehicle's resonance condition is met. Third, the feedback effect of the bridge's resonance to the vehicle's response is large, but the vehicle's resonance effect on the bridge response is quite small, especially along vertical direction. Fourth, no resonance of the middle span of the continuous bridge occurs for the second mode in the vertical and radial directions and fifty the increase in the number of spans result in both smaller displacement and lower impact factor.

Xia et al., (2015) have based their study on the theoretical solution for vibrations of simply supported bridges under moving train loads and derived resonance and cancellation. When cancellation occurs, the free vibrations induced by the moving loads cancel to null and the bridge response is determined by the load still on it. A resonance disappearance effect occurs when train speed meets both the resonance and cancellation conditions, while cancellation plays a predominant role. Bridge damping has an influence on the cancellation effect: the higher the damping and the longer the interval between the loads, the lower is the cancellation efficiency. When a bridge is subjected to a moving load series, the resonance of the bridge caused by the accumulation of free vibrations is more detrimental than the one appearing in the forced-vibration stage because each forced vibration is related to only a half-period of the fundamental bending mode of the bridge, while the speed associated with the second resonant condition is much higher than the current operational train speed. Therefore, the first resonant condition is of concern in the dynamic design of high-speed railway bridges.

Xia et al., (2014) studied vibrations of the simple beam under a single moving force, equidistant load series and train loads. The first cancellation is simply associated with the moving speed of a single force; the second one takes place between two loads and gets their interval involved. Resonance will be suppressed and phenomenon of resonance disappearance is expected when conditions of resonance and vibration cancellation met simultaneously. Museros & Martínez-Rodrigo [9] in their papers they found out that both resonance and cancellation situations are related to the free vibration response of beams but the former needs a repetitive loading pattern to develop while the latter takes place with the passage of every single load. The previously mentioned works are dedicated to the analysis of beams with simply supported end conditions. Martinez-Rodrigo et al., (2013), studied free vibrations of simply supported beam bridges under moving load on the resonance phenomena they found out that if resonance speeds coincide with either a maximum free vibration or a cancellation speed then a maximum resonance or a cancellation of resonance will occur.

Yang et al., [16] have investigated the effect of the vertical stiffness of elastic bearings, in the context of High-Speed railway bridges, concluding that it may be of great importance to consider the effect of these elements as they lead to a reduction of the bridge deck natural frequencies, and therefore of the critical or resonant speeds, and to a variation of the resonant amplitudes which is not easy to predict a priori. For this reason, in this study the cancellation and resonance phenomena in elastically supported (E-S) beams is investigated in detail. Yang et al., [16] address this problem approximating the first mode of vibration of an E-S beam by the combination of a flexural sine mode (S-S case) and a rigid vertical displacement mode.

Li & Su [7] divided the bridge vibration under moving loads into two stages: the forced vibration when the load is moving on the bridge and the free vibration after the load has left it. They found out that the bridge response is amplified constantly because of the accumulation of free vibrations induced by a series of loads. Matsuura [11] on his paper revealed that since the construction of the first high-speed railway, the resonant effect has been found in the distribution curves of the dynamic amplification factor (DAF) for several types of bridges. In China, during the dynamic analysis of bridges on the Beijing-Shanghai high-speed railway, the resonance problem was studied for various bridge types considering the loads of the Germany ICE3, French TGV, Japanese E500, and several types of China-made high-speed trains. Yau (2001) studied continuous bridges subjected to the passage of high speed train. Only bridges with uniform spans were considered. He found out that the more the number of spans of continuous beam, the smaller the impact is for the displacement. Yang et al., [15] studied resonance conditions by modeling a train as two sets of axle loads with constant intervals, and discovered that resonance of the bridge may not occur at certain ratios between the span length and characteristic load distance.

Pesterev et al., [12] discovered that for a single load passing a simply supported bridge at various speeds, the amplitude of the free vibration changes and declines to null at some speeds. This interesting finding explains the vibration cancellation effect suggested by Yang et al., [15] and Savin [13], namely, when a single load passes the bridge at a cancellation speed, the residual free vibration of the bridge will be null after the load leaves it. As a further study, Yang et al., [16] and Yau et al., (2001) investigated the resonance and the cancellation effect on the fundamental bending mode of a simply supported bridge with elastic supports. Unlike the resonance effect that enlarges the bridge response, the cancellation effect may suppress the vibration of the bridge, which is favorable for train running safety and bridge service. Therefore, a further investigation of the cancellation effect is of significant importance.

Euler-Bernoulli beam equation (2009) describes the relationship between the beam deflection and applied load as:

$$\partial^2 / \partial x^2 \left(EI \frac{\partial^2 u}{\partial x^2} \right) = w \qquad (2)$$

where the curve $u(x)$ describes the deflection of the beam at some position x, w is distributed load in other words force per unit length. E is the elastic modulus and I is the second momentum of area. For Euler –Bernoulli's not under axial loading this is called neutral axis. Often $u = u(x)$, $w = w(x)$, EI is a constant so that,

$$\frac{d^4 u}{dx^4} EI = w(x) \qquad (3)$$

This equation describes the deflection of a uniform; static beam is very common in engineering practice. According to Singh [14] used Rayleigh method has been used to approximate the deflection $u(x)$ of a simply supported beam as follows. Consider a load distribution $q(x)$ applied to a beam. If dx element is isolated at a position x, the net force on the element is qdx. The beam deflects in the positive direction in a linear fashion a distance $u(x)$, the net work performed is $-\frac{1}{2} qu(x)dx$. The work is negative since the force and deflection are in opposite direction. The shape of the deflection curve is a function of x and consistent with geometric boundary conditions. According to Zafer [22] Laplace transform method gives the deflection of a simply supported beam L carrying a uniformly distributed load (q) on the central half length (L).

Lazer & McKenna [5] proposed a nonlinear beam equation as a model for vertical oscillations in suspension bridges. They modeled the restoring force from the cable as a piecewise linear function of the displacement in order to capture the fact that the suspension cables resist elongation, but do not resist compression. Later investigations of the qualitative and quantitative properties of solutions to this type of asymmetric system suggest that this is a convincing model for nonlinearly suspended structures.

A few researchers have studied this phenomenon. Xia and Roeck (2014) looked at the theoretical solution for vibrations of simply supported bridges under moving train loads, where resonance and cancellation conditions are derived. Maria et al., (2010) investigated the dynamic behavior of beams leaning on identical elastic supports subjected to the circulation of moving loads at constant speeds.

From the view of dissipating the vibration energy, Hwang & Tseng [4] conducted the seismic response control of highway bridges installed viscous dampers and proposed design formulations of supplemental linear and non-linear viscous dampers. Recently, Reis & Pala [17] studied the response of a simply supported beam with viscous damping carrying a moving force by Fourier sine series approach. Yang et al., [20] investigated the performance of a tuned mass (TM) (with no damping capability) in suppressing the vibration response of an elastically supported beam to a moving train.

Lou [8] derived a finite train-track-bridge coupling element with train and track contact model for analyzing dynamic responses of train, track and bridges. Didn't considering the separation of train and bridge, Ziyaeifar [21] proposed a Maxwell (three-element type) vehicle-bridge track interaction system and studied the effects of TMD devices in vibration control of bridges and trains. Xia & Zhang [18] built a 3D train-bridge dynamic model to study the dynamic interaction between high-speed trains and bridges. In this model, the vehicle was modeled by the rigid-body dynamics method and the bridge was modeled using the modal superposition technique. Besides, Xia et al., [19] also investigated the resonance mechanism and conditions of the train-bridge system through theoretical derivations, numerical simulations, and experimental data analyses. Lee & Kim [6] also assumed the wheel remained in contact with the bridge and developed 3D dynamic analysis model to predict accurately the dynamic response of a high speed train interacting with a railway bridge. He et al., [3] modeled the carriage as 15 DOFs sprung-mass system and the bridge with 3D finite elements, and evaluated the influence of dynamic bridge-train interaction (BTI) on the seismic response of the Shinkansen system in Japan under moderate earthquakes.

From the review of the pertinent literature presented above, it is seen that elastically supported beams with application to excited bridge vibrations has received considerable attention. However, there exists scope for further investigation on analysis of the effects of the cross sectional area and the external load on the transverse displacement of a beam on excited bridges numerically using of Finite Difference Method.

1.6. Statement of the Problem

Suspension bridges are generally susceptible to visible oscillations or vibrations, which if not controlled can lead to failure of the bridge. Beams experience compressive, tensile and shear stresses as a result of the loads applied to them. The possibilities to predict such situations could be of special interest to researchers in an attempt to provide solutions to High-Speed road and railway bridge failures. Therefore, there is need to study factors that cause effects on transverse displacement of a beam on excited bridges. Therefore the factors studied in this study include; the effect of cross sectional area and the external load on the transverse displacement of a beam on excited bridges.

1.7. General Objective

The objective of this study is to investigate numerically effects of some parameters on vibrations of elastically supported beam with application to vibrating bridges.

1.7.1. Specific Objectives of the Study

The specific objectives of the study are:

i) To determine the effects of the cross sectional on the transverse displacement of a beam on excited bridges.
ii) To determine the effects of the external load on the transverse displacement of a beam on excited bridges.

1.8. Justification of the Study

Many bridge failures have occurred world over. It is therefore necessary to study the effects of the cross sectional area and also the effects of the external load on the transverse displacement of a beam on excited bridges. This may benefit the following groups; the government benefit by minimizing bridge failures by the use of commendable material, civil and structural engineers will find a solution to the otherwise unpredictable bridge failures and the study will also give a mathematical contribution to the field.

It is necessary to study the principles of Beam equation in order to solve vibration problems involving structural mechanics. The knowledge from this study useful to civil engineers, structural engineers and also contribute to mathematical knowledge in this field.

In the next section the governing equations are discussed.

II. GOVERNING EQUATIONS

2.1. Bernoulli-Euler Beam Equation Non-Excited Vibrations

Beam equations have a long history starting from Leonardo da Vinci (1452-1519) and Galileo Galilee (1584-1642) developed by Leonard Euler (1707-1783), Jacob (1654-1705) and Daniel Bernoulli (1700-1782) in the eighteenth century. There are numerous structures that evidently were constructed through application of the beam theory. Practical applications of the beam equations are presented in broad by Clough & Penzien [1], Gere [2] and Popov [10]. These are very huge buildings like the Manhattan towers, long bridges across big rivers, aero planes and cars. In these structures, the beams are either used as supporting structures of the floor or as axles of cars.

The non-excited vibration of a Bernoulli-Euler beam equation of motion with length L has the Form;

$$EI\frac{\partial^2 u}{\partial t^2}+\rho A\frac{\partial^4 u}{\partial x^4}=0 \tag{4}$$

2.2. Bernoulli-Euler Beam Equation for Excited Vibrations

The natural way for describing the external excitation in equation of motion of the Bernoulli-Euler type beam is putting the loading function in a form of linearly distributed force $q(x;\, t)$. In realistic application sometimes is better to model the external force in a form of set of concentrated forces The problem of excited vibrations of the Bernoulli-Euler beam with ends supported is modeled by the equation;

$$EI\frac{\partial^2 u}{\partial t^2}+\rho A\frac{\partial^4 u}{\partial x^4}=f\left(x,t\right) \tag{5}$$

where $u(x;\, t)$ is the transverse displacement of a beam E is the Young modulus of material, ρ is the volume density of material, I is the moment of inertia of the beam cross-section, A is the area of the beam cross-section and $f(x.t)$ is the external forcing term. $E\times I$ is the product of Young's modulus of elasticity E and moment of inertia I of the beam. It is referred to as the flexural rigidity, and is a measure of strength.

2.3. Discretization of Equation (5)

Discretization of Equation (5) is obtained by replacing partial derivatives appearing in the equation with their Finite difference approximations as follows

$$\frac{\partial^2 u}{\partial t^2} = \frac{U_{i,j}^{n+1} - 2U_{i,j} + U_{i,j}^{n-1}}{(\Delta t)^2} \tag{6}$$

$$\frac{\partial^4 u}{\partial x^4} = \frac{U_{i,2j} - 4U_{i+1,j} + 6U_{i,j} - 4U_{i-1,j} + 4U_{i-2,n}}{(\Delta x)^4} \tag{7}$$

In section three the method of solution is discussed.

III. Method of Solution

3.1. Computational Procedure

In this study a Hybrid numerical scheme was developed and Finite Difference Method was used solve the governing equations. We solved the governing equations subject to the given boundary conditions. MATLAB software was used to generate solution values in this study.

3.2. Discretization of the Governing Equations

In this section, equation governing bridge vibrations are discretized. Using Central Difference numerical scheme, u_{tt} and u_{xxxx} are replaced by central difference finite difference approximations in equation (6) and (7) respectively, then equation (5) becomes

$$El\left[\frac{U_{i,2j} - 4U_{i+1,j} + 6U_{i,j} - 4U_{i-1,j} + 4U_{i-2,n}}{(h^4)}\right] + \rho A\left[\frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{k^2}\right] = f(x,t) \tag{8}$$

3.3. Effects of External Force $f(x,t)$ on Bridge Transverse Vibrations

We investigate the effect of $f(x,t)$ on the bridge vibrations. Taking mesh sizes of $\Delta x = \Delta t = 0.1$ these sizes test convergence but lesser sizes may be more accurate, $A = 40m^2$ where it is assumed the cross section is $4\ m$ by $10\ m$ and assuming the bridge is made of concrete and steel, the flexural rigidity $El = 80N/mm^2$ and material density $\rho = 15\ kN/mm^2$ respectively, the average load of the vehicles is taken from a minimum of 5 tons $f(x,t) = 50{,}000N$ substituting these values in equation (11), we get the scheme below.

Taking and $i = 1,2,3 \ldots\ldots 12$ and $j = 1$ we form the following systems of linear algebraic equations

$$\left.\begin{array}{l}
-320U_{i,j+1} + 477U_{i,j} - 320U_{i-1,j} = -81.5U_{i,j+1} - 80U_{i-2,j} - 1.5U_{i,j-1} + 0.5 \\
-320U_{i,j+1} + 477U_{i,j} - 320U_{i-1,j} = -81.5U_{i,j+1} - 80U_{i-2,j} - 1.5U_{i,j-1} + 0.5 \\
-320U_{i,j+1} + 477U_{i,j} - 320U_{i-1,j} = -81.5U_{i,j+1} - 80U_{i-2,j} - 1.5U_{i,j-1} + 0.5 \\
-320U_{i,j+1} + 477U_{i,j} - 320U_{i-1,j} = -81.5U_{i,j+1} - 80U_{i-2,j} - 1.5U_{i,j-1} + 0.5 \\
-320U_{i,j+1} + 477U_{i,j} - 320U_{i-1,j} = -81.5U_{i,j+1} - 80U_{i-2,j} - 1.5U_{i,j-1} + 0.5 \\
-320U_{i,j+1} + 477U_{i,j} - 320U_{i-1,j} = -81.5U_{i,j+1} - 80U_{i-2,j} - 1.5U_{i,j-1} + 0.5 \\
-320U_{i,j+1} + 477U_{i,j} - 320U_{i-1,j} = -81.5U_{i,j+1} - 80U_{i-2,j} - 1.5U_{i,j-1} + 0.5 \\
-320U_{i,j+1} + 477U_{i,j} - 320U_{i-1,j} = -81.5U_{i,j+1} - 80U_{i-2,j} - 1.5U_{i,j-1} + 0.5 \\
-320U_{i,j+1} + 477U_{i,j} - 320U_{i-1,j} = -81.5U_{i,j+1} - 80U_{i-2,j} - 1.5U_{i,j-1} + 0.5 \\
-320U_{i,j+1} + 477U_{i,j} - 320U_{i-1,j} = -81.5U_{i,j+1} - 80U_{i-2,j} - 1.5U_{i,j-1} + 0.5 \\
-320U_{i,j+1} + 477U_{i,j} - 320U_{i-1,j} = -81.5U_{i,j+1} - 80U_{i-2,j} - 1.5U_{i,j-1} + 0.5 \\
-320U_{i,j+1} + 477U_{i,j} - 320U_{i-1,j} = -81.5U_{i,j+1} - 80U_{i-2,j} - 1.5U_{i,j-1} + 0.5
\end{array}\right\} \tag{10}$$

The above algebraic equations can be written in matrix form as when $U(0,t) = 0$ and $U(x,0) = 0$.

$$\begin{bmatrix}
477 & -320 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-320 & 477 & -320 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -320 & 477 & -320 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -320 & 477 & -320 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -320 & 477 & -320 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -320 & 477 & -320 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -320 & 477 & -320 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -320 & 477 & -320 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -320 & 477 & -320 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -320 & 477 & -320 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -320 & 477 & -320 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -320 & 477
\end{bmatrix}
\begin{bmatrix} U_{1,1} \\ U_{2,1} \\ U_{3,1} \\ U_{4,1} \\ U_{5,1} \\ U_{6,1} \\ U_{7,1} \\ U_{8,1} \\ U_{9,1} \\ U_{10,1} \\ U_{11,1} \\ U_{12,1} \end{bmatrix}
=
\begin{bmatrix} -0.5 \\ -0.5 \\ -0.5 \\ -0.5 \\ -0.5 \\ -0.5 \\ -0.5 \\ -0.5 \\ -0.5 \\ -0.5 \\ -0.5 \\ -0.5 \end{bmatrix} \tag{11}$$

Solving the above matrix equation (11) using MATLAB, we get the solutions for changing $f(x,t)$.

3.4. Effects of Cross-Sectional Area on Bridge Transverse Vibrations

We investigate the effect of cross-sectional area A, on the bridge vibrations. Taking $\Delta x = \Delta t = 0.1$, $A = 40m^2$, $El = 80N/mm^2$, $f(x,t) = 50{,}000N$ and $\rho = 15\ kN/mm^2$ we.

Using the scheme in (12) above, we get the solutions for changing A as follow.

IV. Results and Discussion

4.1. Introduction

The simulated results shows relationships between bridge transverse vibrations and various parameters (i.e force term $f(x,t)$ and cross sectional area A) as obtained by numerical computation are given in Figs 2a, 2b, 3a and 3b below. The simulation results given focus on the effects of the external force $f(x,t)$ and cross-sectional area A, on bridge transverse vibrations.

4.2. Effects of External Force $f(x,t)$ on Bridge Transverse Vibrations

The results obtained after solving matrix equation (11) using MATLAB for the effects of the bridge vibrations are shown in table 1 below.

Table 1: Values Bridge Transverse Vibrations for Varying External Force $f(x,t)$

Bridge Length (L)	f(x,t)=50,000N	f(x,t)=100,000N	f(x,t)=150,000N	f(x,t)=200,000N
0	6.617223×10^{-2}	0.1323467	0.198 332224	0.266544
1	1.002214×10^{-1}	0.267489	0.36755542	0.4544327
2	8.472768×10^{-2}	5.544582×10^{-2}	0.2534222	0.3397655
3	2.773574×10^{-1}	-0.1232566	8.3433221×10^{-2}	0.11 43877
4	-1.674×10^{-1}	-0.26589544	-8.3825488×10^{-2}	-4.19432×10^{-2}
5	-3.5245×10^{-1}	-0.2653427	-0.177 23728	-8.86896×10^{-2}
6	-3.52134×10^{-1}	-0.1251248	-0.177 23728	-8.86896×10^{-2}
7	-1.67986×10^{-1}	5.54663×10^{-2}	8.3433221×10^{-2}	-4.19432×10^{-2}
8	2.773574×10^{-2}	0.2977433	8.3433221×10^{-2}	0.11 43877
9	8.472315×10^{-2}	0.169658941	0.2534222	0.3397655
10	1.002215×10^{-1}	0.1343688	0.36755542	0.4544327
11	6.622875×10^{-2}	0.1323467	0.198 332224	0.266544

Table 1 is used to analyze two parameters namely; Transverse bridge vibrations for varying external force $f(x,t)$ and Transverse bridge vibrations against bridge length.

The results in the table 1 above is represented graphically as seen in figures 2a and 2b below

When the columns of table 1 above are considered, we plot the values of transverse vibrations $u(x,t)$ against bridge length L or x. at varying external force $f(x,t)$. This gives the graph 2a below.

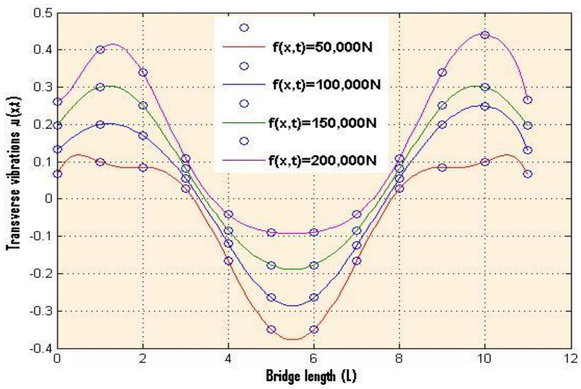

Figure 2a: Graph of Transverse Vibrations against Bridge Length at Varying External Force

The graph clearly shows that an increasing in the load also increases the bridge transverse vibrations at each point on the bridge. It should also be noted that the bridge transverse vibrations for the simply-supported bridge is greater when external force on it increases.

It can be seen that at mid-span, by symmetry of the beam and loading, the slope of the curve which is the term dv/dx must be zero. In practice it is the maximum deflection that is of interest and common sense would say that this occurs at mid-span. If it is not obvious where the maximum deflection occurs it can always be determined by knowing that it will occur where there is a change in slope of the beam i.e. where $dv/dx = 0$.

When the rows of table 1 are considered, the values of $u(x,t)$ are plotted against $f(x,t)$ at varying bridge lengths to get the graph in figure 2b as shown below.

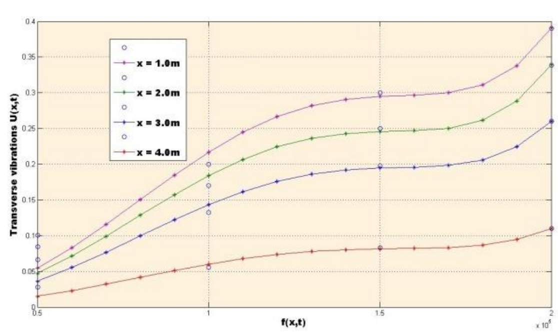

Figure 2b: Graph of Transverse Vibrations against External Force at Varying Bridge Length

Table 1 is also used to draw the graph transverse vibrations of $u(x,t)$ against external force $f(x,t)$ at varying bridge lengths. The effect of the external force on the overall bridge transverse vibrations is depicted in Figure 2b.The steepness of curve in this figure increases as external force is also increases. This is due to the fact that a small force creates a small frequency or oscillation on the bridge while a big force causes large oscillations.

4.3. Effects of Cross-Sectional Area on Bridge Transverse Vibrations

The results obtained when matrix equation (11) is solved using MATLAB for the effects of varying cross sectional area are summarized and shown in table below.

Table 2: Values of Bridge Transverse Vibrations for Varying Cross-Sectional Area

Bridge Length (L)	**A=50m²**	**A=40m²**	**A=30m²**	**A=20m²**
0	8.27×10^{-4}	1.1×10^{-3}	1.4×10^{-3}	1.6×10^{-3}
1	2.48×10^{-3}	2.7×10^{-3}	2.9×10^{-3}	3.1×10^{-3}
2	3.38×10^{-3}	3.6×10^{-3}	3.8×10^{-3}	4×10^{-3}
3	3.67×10^{-3}	3.8×10^{-3}	4×10^{-3}	4.2×10^{-3}
4	3.27×10^{-3}	3.5×10^{-3}	3.7×10^{-3}	3.9×10^{-3}
5	1.59×10	1.9×10^{-3}	2.1×10^{-3}	2.3×10^{-3}
6	2.07×10^{-4}	0.5×10^{-3}	0.7×10^{-3}	0.9×10^{-3}
7	1.07×10^{-4}	0.3×10^{-3}	0.5×10^{-3}	0.7×10^{-3}
8	$2.07\times10^{-}$	0.5×10^{-3}	$0.7V10^{-3}$	$0.9V10^{-3}$
9	1.29×10^{-3}	$1.6*10^{-3}$	1.8×10^{-3}	2.0×10^{-3}
10	3.17×10^{-3}	3.5×10^{-3}	3.7×10^{-3}	3.9×10^{-3}
11	3.67×10^{-3}	3.8×10^{-3}	4.0×10^{-3}	4.2×10^{-3}
12	3.38×10^{-3}	3.6×10^{-3}	3.8×10^{-3}	4×10^{-3}
13	2.48×10^{-3}	2.7×10^{-3}	2.9×10^{-3}	3.1×10^{-3}
14	8.277×10^{-4}	1.2×10^{-3}	1.4×10^{-3}	1.6×10^{-3}

The results in the table 2 above is represented graphically in figure 3a and 3b below

When the columns of table 2 above are considered, we plot the values of transverse vibrations $u(x,t)$ against bridge length L or x at varying cross-sectional area (A). This gives the graph 3a below.

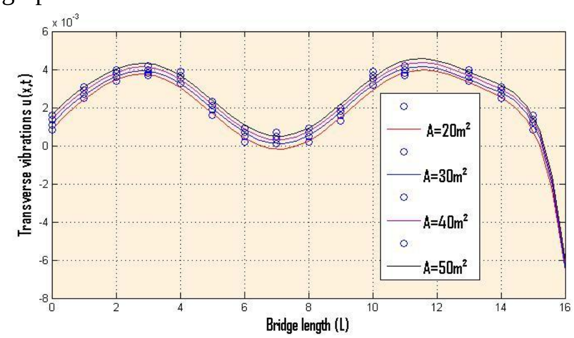

Figure 3a: Graph of Transverse Vibrations against Bridge Length at Varying Cross-Sectional Area

From figure 3a above, it is shown that as the bridge length increases, the transverse vibrations of the bridge fluctuate. The result indicates that the variation in length has a constant effect on the transverse vibrations for all cross sectional areas. Thus, the change in length of the beam has a constant effect on its transverse vibrations for varying cross-sectional area.

When the rows of table 2 are considered, the values of $u(x,t)$ are plotted against cross-sectional area (A) at varying bridge length.

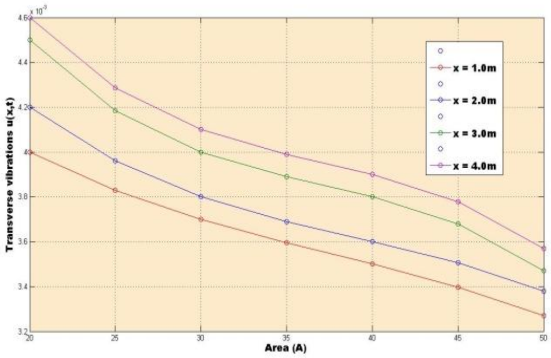

Figure 3b: Graph of Transverse Vibrations against Cross-Sectional Area at Varying Bridge Length

For an increase in the value of cross sectional area, the transverse vibration of the bridge also decreases. The result indicates that variation of the cross sectional area has a constant effect on the transverse vibrations. The change in cross sectional area of the beam has a constant effect on the transverse vibrations of the beam for various boundary conditions and length.

4.4. Discussion

When the length of the bridge becomes greater, it leads to increases of transverse vibrations of the bridge as opposed to shorter lengths.

It is seen than when the cross-sectional area of the bridge is increased, the transverse vibrations of the bridge decreases. This is due to the fact that, if the area is greater, the beam becomes stiffer in bending for a given material and hence the smaller the bridge vibrations. Thin walled bridges exist because their bending stiffness per unit cross sectional area is much higher than thicker walled bridges. In this way, stiff bridge beams can be achieved with minimum weight.

In the next section the conclusion and recommendations of this study are highlighted.

V. Conclusion and Recommendations

5.1. Conclusion

In this study, the transverse vibrations of bridge beams are investigated numerically under specified different boundary conditions. Numerical solution of Euler-Bernoulli beam equation is carried out. The solutions obtained for Euler-Bernoulli beam equation take into consideration the effects of bridge beam material's geometric characteristics, i.e., length and cross sectional area, and boundary conditions. From the results we have found that;

(i) The transverse vibrations of bridge beams increase with increasing bridge length.
(ii) The transverse vibrations of bridge beams increase with decreasing bridge cross sectional area.
(iii) The increase in forcing term on the bridge beam causes an increase on the transverse vibrations of bridge beams.

5.2. Recommendations

The study dwelt on the effects of varying external force and cross sectional area on the transverse vibrations of bridge beams is very important especially in the study of excited bridges. From this study, there are areas that arise for further analysis and development. These may be theoretical or experimental and specific areas of study include:

(i) Investigate the effects of varying density of bridge material on the transverse vibrations of bridge beam
(ii) Solve the model Equation using other numerical and analytical methods and use the results to investigate excited bridge beams

The intention is to carry out further studies on some of the open problems and methods in the above mentioned area of study.

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